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ABSTRACT

This volume contains preliminary versions of five of the chapters prepared by the SMSG curriculum project for use in grades 7 and 8. The first four chapters and the tenth chapter in the sequence are presented. The sample chapters in this volume illustrate a number of aspects of the curriculum project: (1) association of ideas of number and space through coordinate geometry: (2) early introduction of the function concept: (3) development of flow charts and algorithms as an introduction to the role and use of computers in modern society: (4) attention to the role of mathematical models for physical situations: and (5) introduction of concepts of probability. (MP)

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SECONDARY SCHOOL MATHEMATICS

(Preliminary Version)

Sample Chapters

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SECONDARY SCHOOL MATHEMATICS

(Preliminary Version)

Sample Chapters

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PREFACE

The SMSG Newsletter No. 24 (October 1966) contained a Preliminary Report of a New Curriculum Project. The aims and general character of the project are spelled out in some detail in that document.

This curriculum effort is designed to produce an experimental set of materials for grades 7 through 9, which build on a modern elementary school program and take into account recent changes which have occurred in mathematics and in the use of mathematics. A consistent emphasis on the relevance of mathematics to problems of the real world; a rethinking of the ordering of topics and the combining of arithmetic, algebra, and geometry; the use of the function concept whenever appropriate; and the inclusion of material on mathematical models, flow charts, and probability and statistics are some of the features of these materials.

Since the report in the Newsletter appeared, drafts of a substantial number of chapters designed for use in grades 7 and 8 have been prepared. This volume contains preliminary versions of five of these chapters, the first four and the tenth in the sequence as presently conceived. The first four chapters were taught in experimental seventh-grade classes in 1968-69 and are revisions of earlier drafts taught by the same teachers in 1967-68. Chapter 10 is a slightly revised version of an earlier draft taught by the same teachers in eighth-grade classes early in the 1968-69 school year. Further revision of these chapters is anticipated.

The sample chapters in this volume illustrate a number of aspects of the curriculum project as described in Newsletter 24: association of ideas of number and space through coordinate geometry; early introduction of the function concept; development of flow charts and algorithms as an introduction to the role and use of computers in modern society; attention to the role of mathematical models for physical situations; and introduction of concepts of probability.

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Student's Text

Chapter 1

STRUCTURING SPACE

1-1. *Introduction*

Geometry began a few thousand years ago when man began to study systematically the size, shape, and location of objects in his physical surroundings. This understanding was necessary then as it is today in such activities as navigation, building, and surveying. Today, however, we find that geometry has been considerably refined and extended to deal with the more complicated problems which confront us.

Geometry is a system of ideas based on our experiences with physical objects. For example, from our experiences with various boxes we have developed an idea of the shape and form of a box. This idea we have of a box is called a *geometric figure* and it is this figure, not the box which is studied in geometry. In other words, geometry is developed as a *mathematical model* of our experience with physical space.

The idea of a *point* in geometry is formed from such objects as the end of a pin, a speck of dust, a grain of sand, and a distant star. Consequently, we may think of a point as a definite location in space.

Points are represented by dots and are named by capital letters as is shown for point A at the left.

We think of space as being a set of points. There are an unlimited number of points in space. Suppose that a mosquito is flying about inside a room. Its location at a particular instant might be described as a point and all such points make up a set. All the points inside of this room make up a portion or subset of space and all the points outside this room make up another portion or subset of space. Therefore in geometry, space is the set of *all* points.

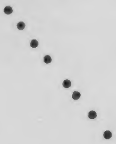
Each of the various kinds of geometric figures studied here will be developed as a set of points. We shall see how space is structured by carefully examining the way in which the various figures are formed and related to each other using the basic ideas of point, line, and plane. Later in this chapter a systematic way of assigning numbers to points in a plane will be developed. This will provide us with another very useful method of studying the structure of space.

Exercises 1-1

(Class Discussion)

1. Complete the following sentence.
   Geometry is the study of ____?____.
2. What is the relationship between geometry and physical objects?
3. Describe three objects which can be used to form the idea of a point, other than the objects already mentioned.
4. What meaning would a point have for a surveyor or navigator? Give some examples.
5. If you were to think of a point as a definite location, how would this affect the way in which you would represent a point?
6. In forming the idea of a box as a geometric figure, what are some things about a box you would ignore?

1-2. Lines and Points



The points shown at the left lie on what we think of as a "straight line", or simply a "line".

There are an unlimited number of points on a line, and a line extends without ending in both directions.

Think of two boys holding a string stretched between them as shown below.

When the string is stretched tight between the boys' hands, the stretched string represents part of a line. The line itself, however, goes beyond the boys' hands without ending.

Some other objects which help to form the idea of a line are

(1) the edge of a ruler,

(2) the crease of a carefully folded paper,

(3) the edge of a box,

(4) a flagpole.

The drawing below representing line ℓ can be made by tracing a pencil along a straightedge on a sheet of paper.

What do the arrowheads on the drawing indicate?

Mark a point P on your paper and draw three lines, ℓ, m, and n which contain point P. Are there other lines on which point P lies?

We can think of many lines passing through a point. For two different points, however, we think of only one line passing through them since we think of a line as being straight. We shall accept the following statement as a fundamental characteristic or property of the structure of our space.

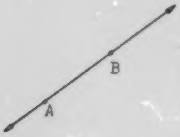
Property 1: Through any two different points in space there is exactly one line.

This property may be restated in the following ways.

(1) For any two points there is one line and only one line containing them.

(2) Any two points determine a line.

As a result of Property 1, a line can be clearly identified by naming any two of its points.



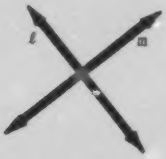
For example, the line shown at the left can be identified as "line AB" which is written "$\overleftrightarrow{AB}$".

It may also be identified as "line BA" and written "$\overleftrightarrow{BA}$".

Exercises 1-2a

(Class Discussion)

1. The following drawing was made on the blackboard with a broad piece of chalk. It represents two lines ℓ and m that cross or intersect.



   (a) How does this drawing suggest the possibility that the intersection of ℓ and m may contain several points?

   (b) Let's explore this possibility by assuming that the intersection of ℓ and m contains two points, A and B.

   Show that this means we have two lines passing through the same two points.

   But Property 1 states that there can be only one line passing through any two points!

   (c) If we wish to continue to include Property 1 in the structure of our space why must we exclude the possibility of there being two points in the intersection of two lines?

   What about the possibility of there being more than two points in the intersection?

We have the following statement as a rather surprising result of Property 1.

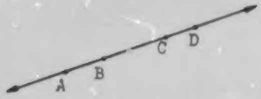
If two lines intersect, they intersect in exactly one point which is called the *point of intersection*.

This statement together with Property 1 can be summarized as follows.

Two points determine a line and two intersecting lines determine a point.

Exercises 1-2b

1. Name three objects other than those already mentioned which suggest the idea of a line.

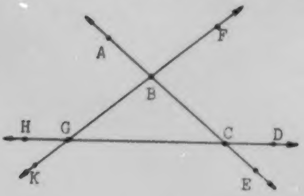
2. 

Points A, B, C, and D lie on the same line.

(a) Using these points, write five different names for this line.

(b) How many different names are there using these points?

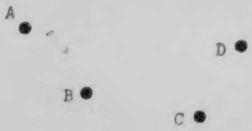
3. (a) How many lines are shown below?



(b) Name the lines.

(c) Name the point of intersection for each pair of lines.

4. Copy the set of points {A,B,C,D} arranged as shown.



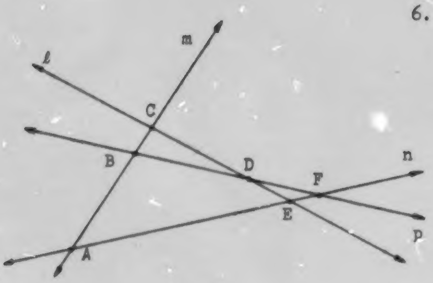
(a) Draw all the lines you can through pairs of these points.

(b) Name the lines you have drawn.

(c) How many such lines are there?

5. Think of 5 points marked on a sheet of paper, no three of which lie on the same line. Now think of all the lines through pairs of these points.

How many such lines are there?
Check your answer with a drawing.



6. Name the point of intersection for the following pairs of lines.

   (a) m and p

   (b) p and ℓ

   (c) ℓ and n

7. For the figure in Exercise 6, name the pairs of lines which have the following points as points of intersection.

   (a) A

   (b) C

   (c) F

1-3. Planes

A plane is an idea in geometry which is suggested by any flat surface such as

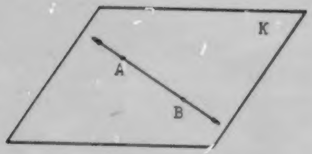
(1) a wall of a room,

(2) the top of a desk,

(3) a door in any position,

(4) a sheet of paper.

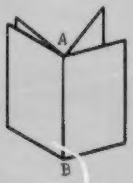
A plane is thought of as extending indefinitely in every direction. There are an unlimited number of points in a plane and a plane contains an unlimited number of lines.



This drawing represents a plane K which contains two points A and B.

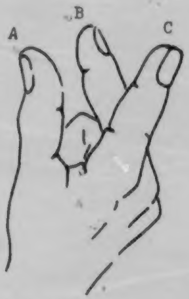
Now consider the line $\overleftrightarrow{AB}$. Both the line and the plane extend outward indefinitely. Because of our thinking that all lines are straight and all planes are flat, we have the feeling that the line $\overleftrightarrow{AB}$ and the plane K will never leave each other. Therefore, we shall agree that the following statement is a fundamental characteristic of our space.

Property 2: If two points of a line lie in a plane then all the points on the line lie in the plane.



The figure at the right represents an open notebook. Points A and B mark the endpoints of the binding, and the pages represent planes. This figure suggests that there are many planes which contain any two points.

Property 3: For any two points in space, there are many planes which contain them.



Now consider three points A, B, and C represented by the tips of your thumb and first two fingers spread stiffly as shown so that the points do not lie on the same line.

Take a piece of cardboard representing a plane and place it on the tips B and C of your two fingers. Show that the cardboard may be rotated in different positions while held fixed to the two finger tips. What property does this show?

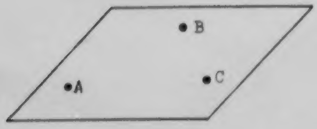
Press the cardboard against all three tips A, B, and C. The position of the cardboard is now fixed. This illustrates another important property.

Property 4. Through any three points, not all on the same line, there is exactly one plane.

This property may also be stated in the following ways.

(1) For any three points in space not all on the same line, there is one plane which contains them and only one plane which contains them.

(2) Any three points not all on the same line determine a plane.



As a result of Property 4, a plane may be named by identifying three of the points in the plane that do not all lie on a line.

For example, the plane shown at the left may be called "plane ABC" or "plane BAC". The three points may be written in any order.

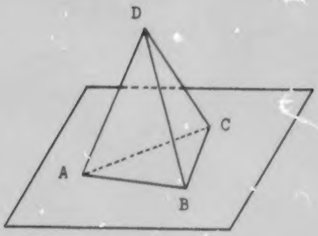
Exercises 1-3

1. Show how Property 4 helps to explain why a three-legged stool or a tripod always rests flat on the floor.

2. Can a line pass through
   (a) any one point?
   (b) any two points?
   (c) any three points?
   (d) any four points?

3. Can a plane pass through
   (a) any one point?
   (b) any two points?
   (c) any three points?
   (d) any four points?

4.



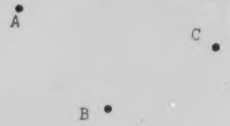
Points A, B, and C lie in a plane and point D is above the plane.

How many planes are determined by these four points?

Name the planes.

5. Copy the points A, B, C arranged as shown.

(a) Draw a line through points A and B. Draw a line through points B and C. Are $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ in plane ABC? Give a reason for your answer.



(b) Mark a point R on $\overleftrightarrow{AB}$ and a point S on $\overleftrightarrow{BC}$. Draw $\overleftrightarrow{RS}$. Is $\overleftrightarrow{RS}$ in plane ABC? Give a reason for your answer.

6. (a) Points X, Y, and Z have the property that many different planes contain them. How are points X, Y, and Z arranged? Draw a diagram.

(b) If points X, Y, and Z have the property that they are contained in only one plane, how are points X, Y, and Z arranged? Draw a diagram.

7. How many different lines may pass through

(a) one point?

(b) two different points?

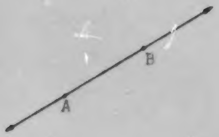
8. How many different planes may pass through

(a) one point?

(b) two points?

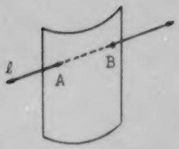
(c) three points which are not on the same line?

9. Explain why there are many planes which may contain a given line $\overleftrightarrow{AB}$.



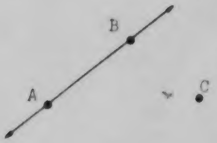
10. Line ℓ intersects the surface shown at points A and B only.

Explain why the surface could not be a plane.



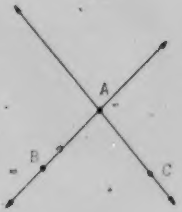
11. Explain why a line $\overleftrightarrow{AB}$ and a point C not on the line determine a plane.

Name the plane.



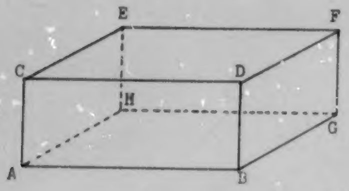
12. Explain why two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ intersecting at A determine a plane.

Name the plane.



13. The drawing below represents a box. Look at the side of the box determined by points B, G, and F.

(a) Name another point that lies in plane BGF.



Name the plane suggested by

(b) the top of the box,
(c) the bottom of the box,
(d) the front of the box,
(e) the back of the box.

1-4. Intersections

Let A and B be names for the following sets.

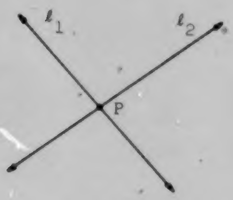
$$A = \{1,3,5,7,9,11\}$$

$$B = \{1,4,9,16,25,36\}$$

The intersection of A and B is the set of all numbers that belong to both A and B. The intersection is the set $\{1,9\}$.

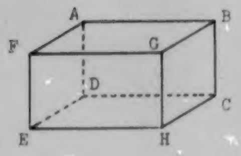
We write: $A \cap B = \{1,9\}$

where "$A \cap B$" means "the intersection of A and B".



The intersection of the lines ℓ_1 and ℓ_2 is the set of all points common to both lines. Since P is the only point in the intersection, we write $\ell_1 \cap \ell_2 = \{P\}$, where "$\{P\}$" means "the set consisting of the point P".

When two lines intersect they have one point in common as was shown from Property 1. When two lines do not intersect they have no points in common.



In the drawing of the box at the left the two lines $\overleftrightarrow{FG}$ and $\overleftrightarrow{EH}$ do not intersect. We say that their intersection is the empty set. The empty set may be named by the symbol "$\emptyset$" or "$\{\}$". Therefore $\overleftrightarrow{FG} \cap \overleftrightarrow{EH} = \emptyset$. Since the intersection of $\overleftrightarrow{FG}$ and $\overleftrightarrow{EH}$ is empty and they lie in the same plane, they are said to be parallel. This is written, $\overleftrightarrow{FG} \| \overleftrightarrow{EH}$, where the symbol "$\|$" means "is parallel to".

The intersection of $\overleftrightarrow{FG}$ and $\overleftrightarrow{HC}$ is also empty, but the two lines do not lie in the same plane. The lines are not parallel but are called skew lines.

When we say that two lines are parallel, we mean:

(1) the two lines lie in the same plane, and

(2) their intersection is the empty set.

Exercises 1-4

1. In each case, write the intersection of the given sets

   (a) A = {2,4,6,8,10}
   B = {1,4,7,10,13,16}

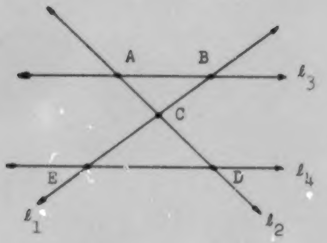
   (b) P = {John, Ethel, Bill, Frank, Alice}
   Q = {Frank, Paul, Alice, Diane, John, Helen}

   (c) The letters used in your full name and the letters used in the name of your school.

   (d) The set of national holidays and the set of days in July.

   (e) The set of school days in this year and the set of Sundays in this year.

2. In the drawing below, $\ell_3 || \ell_4$. Write the intersections of the given pairs of sets.



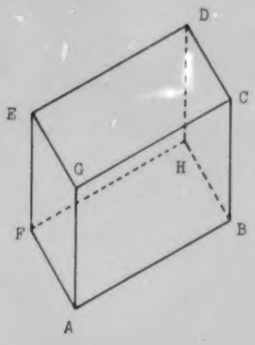
(a) $\ell_1 \cap \ell_3$

(b) $\ell_2 \cap \ell_4$

(c) $\ell_4 \cap \ell_3$

(d) $\ell_1 \cap \ell_2$

3. In the drawing of the box below,



(a) name three lines that are each skew to $\overleftrightarrow{AB}$,

(b) name three lines that are each parallel to $\overleftrightarrow{GC}$.

4. Find the intersections for the following pairs of lines.

(a) $\overleftrightarrow{CD} \cap \overleftrightarrow{HD}$

(b) $\overleftrightarrow{GC} \cap \overleftrightarrow{FA}$

(c) $\overleftrightarrow{BA} \cap \overleftrightarrow{FA}$

(d) $\overleftrightarrow{EH} \cap \overleftrightarrow{BH}$

5. If two lines do not intersect, are they parallel? Explain.

6. Two lines sometimes intersect and sometimes they don't. Also, two lines sometimes lie in the same plane and sometimes they don't.

What do we call two lines which

(a) lie in the same plane and which intersect?

(b) lie in the same plane and do not intersect?

(c) do not lie in the same plane and do not intersect?

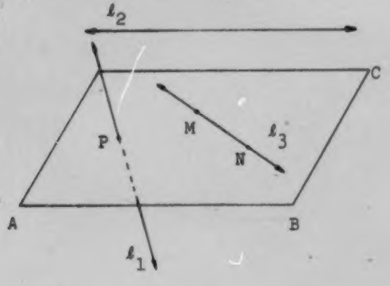
(d) Is it possible for two lines to intersect and not to lie in the same plane?

(e) Copy the table below and write your answers to (a), (b), (c), and (d) in the boxes indicated.

	Intersect	Do not intersect
Lie in same plane	(a) ?	(b) ?
Do not lie in same plane	(d) ?	(c) ?

1-5. Intersections of Lines and Planes

The drawing below shows the possible intersections of a line and a plane ABC. Line ℓ_1 intersects the plane at point P. Then $\ell_1 \cap$ plane ABC = $\{P\}$. Line ℓ_2 is parallel to the plane. Therefore $\ell_2 \cap$ plane ABC = $\emptyset$. Two points M and N of line ℓ_3 lie in the plane. Why does ℓ_3 lie in the plane? Explain why the intersection of ℓ_3 and the plane is ℓ_3 itself. Then $\ell_3 \cap$ plane ABC = ℓ_3.



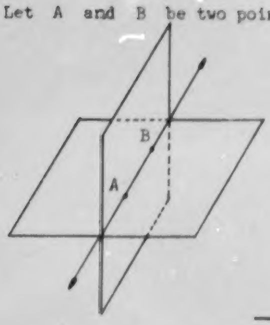
Exercises 1-5a

(Class Discussion)

1. Hold two sheets of paper so that they have only one point of intersection.

   Does this show that two planes may have only one point in their intersection? (Remember that planes extend without limit in all directions.)

2. Let A and B be two points in the intersection of two planes.



(a) Explain why the intersection must contain the entire line $\overleftrightarrow{AB}$.

(b) Explain why if the intersection contains any other point not on $\overleftrightarrow{AB}$ then the intersecting planes are the same plane.

This discussion leads us to another fundamental property in the structure of our space.

Property 5: If the intersection of two different planes is not empty, then the intersection is a line.

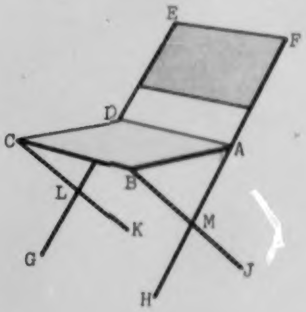
When we say that two planes are parallel, we mean that: their intersection is the empty set.

Exercises 1-5b

1. Describe some physical objects which suggest

   (a) two planes intersecting in a line.
   (b) a line and a plane intersecting in a point.
   (c) two parallel planes.
   (d) the intersection of a plane and a line lying in the plane.
   (e) a line parallel to a plane.

2. Think of the planes and lines suggested by the figure which represents a folding chair.



Write the following intersections.

(a) $\overleftrightarrow{FH} \cap \overleftrightarrow{BJ} =$

(b) $\overleftrightarrow{GL} \cap \overleftrightarrow{AD} =$

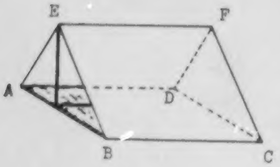
(c) plane ABC $\cap$ $\overleftrightarrow{MJ} =$

(d) $\overleftrightarrow{BC} \cap$ plane EFA $=$

(e) plane ABC $\cap$ plane HMJ $=$

(f) plane CDL $\cap$ plane HMB $=$

3. Think of the lines and planes suggested by this drawing of a tent.



Name

(a) three planes.

(b) three lines which are skew to $\overleftrightarrow{AB}$.

(c) two lines which are parallel to $\overleftrightarrow{BC}$.

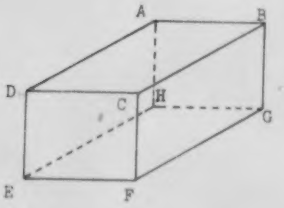
(d) two parallel planes.

(e) two planes whose intersection is $\overleftrightarrow{CD}$.

(f) a line and a plane whose intersection contains the single point E.

(g) a line and a plane that are each parallel to $\overleftrightarrow{AD}$.

4. From the drawing of a box, write the following intersections.



(a) plane EDC ∩ plane FGB =

(b) plane CBG ∩ $\overleftrightarrow{AB}$ =

(c) $\overleftrightarrow{AG}$ ∩ plane FCG =

(d) plane GAB ∩ $\overleftrightarrow{CH}$ =

(e) plane CDG ∩ plane FGB =

(f) plane EDA ∩ plane GBC =

(g) plane FED ∩ $\overleftrightarrow{EC}$ =

5. For the box shown above in Exercise 4, name pairs of planes which have the following lines as lines of intersection.

(a) $\overleftrightarrow{HG}$ (c) $\overleftrightarrow{CE}$

(b) $\overleftrightarrow{FG}$ (d) $\overleftrightarrow{HC}$

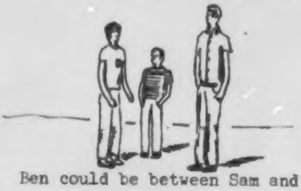
6. Describe the three possible positions for a line and a plane, and in each case tell what their intersection is.

7. Describe the two possible positions for two planes and tell what their intersection is in each case.

1-6. Betweenness and Segments

Three boys are standing together. You're told that, "Tom is tall, Sam is short, and Ben is between them."

Can you tell which boy is named Ben?



The word "between" is confusing here. Ben could be between Sam and Tom in height, or he could be between them in position. The best we could do is guess which boy is Ben.

In mathematics we must not use language which forces us to guess about the meaning of a statement. We need to know exactly what it means to say "point B is between points A and C". We would probably all agree that the phrase should not refer to the alphabetical order of the points because the letters are only names for the points. However, it should have something to do with some kind of order.

We will use the word "between" for points only when the points in question are on the same line. Look at points P, Q, R, and S:

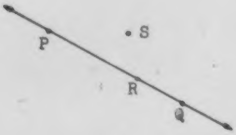


Figure 1

R is on the line $\overleftrightarrow{PQ}$ and the order of the points is P, R, Q (or Q, R, P) so we can say "R is between P and Q". However, point S is not on line $\overleftrightarrow{PQ}$, so it is not between P and Q. Look at the points A, B, X, and Y on line AY:

Is B between A and X?
Is X between Y and A?
Is A between B and Y?

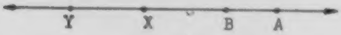


Figure 2

Your answer to the third question should be "No" because the order of the three points involved is Y, B, A (or A, B, Y) and that means point B is the one in between.

In Figure 2 we know that X is between A and Y and that B is also between A and Y. In fact, there are many points between A and Y. We haven't named them all because we couldn't.

When we say that a point P is between points A and B, we mean exactly two things:

(1) There is a line containing A, B, and P;

(2) On that line, the points are in the order A, P, B or B, P, A.

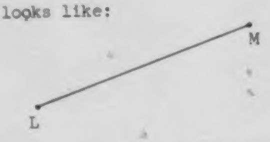
We can simplify this definition a little by using the word "collinear". A set of points is *collinear* if there is a line which contains all the points of the set. For example, in Figure 1 the set of points containing P, Q, and R is collinear, while points P, Q, R, and S are *not* collinear. Using this word, we may restate our definition:

When we say that *point P is between points A and B,* we mean:

(1) *A, B, and P are collinear*, and

(2) *on their line the points are in the order A, P, B or B, P, A*.

Think of two different points L and M. Can you count *all* the points *between* L and M? Think of the two points L and M together with all the points between them. The set of points you are thinking of is a part of a line, and a picture of it looks like:



Such a set of points is called a *line segment*, or more simply, a *segment*. Points L and M are the *endpoints* of the segment. We name the segment by its endpoints; for example, the segment shown is named $\overline{LM}$ or $\overline{ML}$.

Definition. The segment $\overline{AB}$ is the set of points consisting of the two points A, B, and all points between A and B.

Exercises 1-6

1. In the figure at the right,

   (a) is M between B and R?

   (b) is C between B and R?

   (c) is C between M and K?

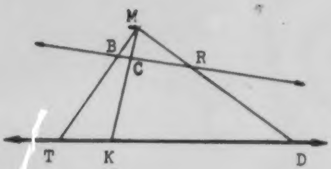
   (d) is C on segment $\overline{MK}$?

   (e) Name 3 segments on $\overleftrightarrow{TD}$.

   (f) Are T, K, and R collinear?

   (g) Is K between T and R?

   (h) Name 4 sets of 3 collinear points.



2. (a) Draw a picture representing 3 segments, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, where points A, B, and C are not collinear.

   (b) What would your picture represent if A, B, and C were collinear?

3. What word in the definition of segment shows that a segment is part of a line? Explain.

4. Points A, B, and C are collinear.
   If the distance from A to B is 5 inches, and
   the distance from B to C is 12 inches, and
   the distance from A to C is 7 inches,
   draw a diagram showing A, B, and C in their correct order. Which point is between the other two?

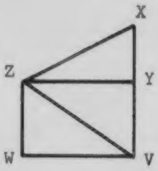
5. In the figure at the right,

   (a) Name 4 segments that intersect at Z.

   (b) How many segments have Y as an endpoint?

   (c) There are 4 segments which contain point Y. Name them.

   (d) What is $\overline{WZ} \cap \overline{WV}$?



6. Draw two segments $\overline{AB}$ and $\overline{CD}$ for which $\overline{AB} \cap \overline{CD}$ is empty but $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is one point.

7. Mark two different points, X and Y.

(a) Draw some segments which contain X and Y.
(b) How many segments could contain X and Y?
(c) How many lines could contain X and Y?
(d) How many segments have both X and Y as endpoints?
(e) We have agreed in Property 1 that "through any two different points there is exactly one line". Is it true that "through any two different points there is exactly one segment"?

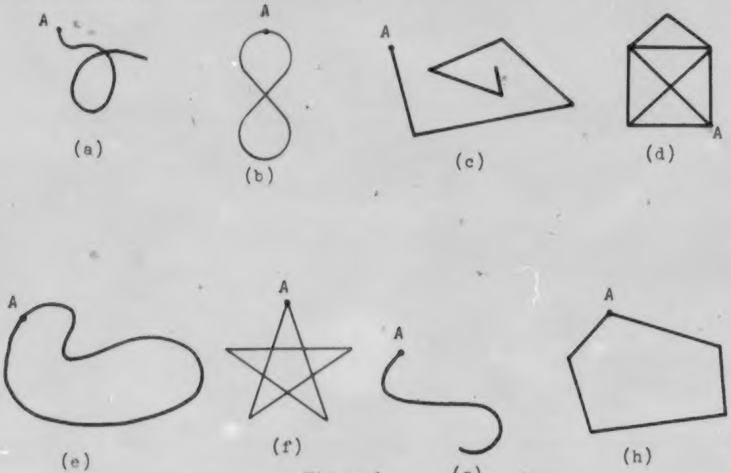
BRAINBUSTER:

Points P, Q, and R are collinear. The distance from P to Q is greater than the distance from Q to R, and the distance from Q to R is greater than the distance from P to R.

Which point must be between the other two?

1-7. Separation

Mark any point A on a sheet of paper, and place the tip of your pencil on that point. Trace out any drawing you like without lifting your pencil from the paper. Your drawing might look like any one of these:



(a) (b) (c) (d) (e) (f) (g) (h)

Figure 1

The geometric idea which all of these pictures represent is called a *path*. The paths you drew were contained in the plane of your paper. It is possible to think of paths in space, but in this section we will work only with paths in a plane.

Look at the paths in Figure 1. Some of them cross themselves, like paths (a), (b), (d) and (f). Some paths do not cross themselves, like (c) and (g). We will call a path which does not cross itself a *simple path*.

Some paths return to their starting points, like (b), (e), (f), and (h). We will call a path which does not cross itself and which goes back to its starting point a *simple closed path*. Which of the paths in Figure 1 are simple closed paths?

What do you suppose a *simple open path* looks like? Such a path does not cross itself and does *not* go back to its starting point. Figures (c) and (g) are examples of simple open paths.

You can invent names for the other special kinds of paths, but we will only be concerned with these two kinds of simple paths.

A simple closed path has an interesting property which no simple open path has. Draw a simple closed path, perhaps something like path p in Figure 2. Mark two points A and B inside the simple closed path, and mark a point C outside.

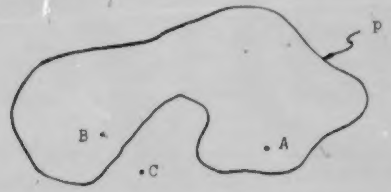


Figure 2

Try to draw a path in the plane from A to B without intersecting p. Try to draw a path from A to C without intersecting p. Finally, try to draw a path from B to C without intersecting p. Were you able to draw all three paths, or were there situations in which you could not?

If you tried to draw all three paths, you probably noticed that a simple closed path like p in Figure 2 has the property that it can *separate* points in the plane. We say that points A and C, for example, are *separated by path p* because (1) A and C are not on path p, and

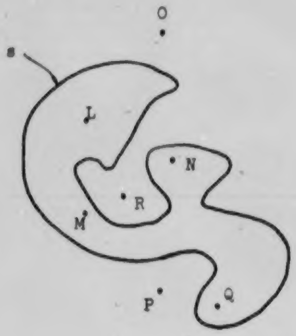
(2) every path connecting A and C intersects p. Notice, however, that points A and B are not separated by path p; you should have been able to draw a path connecting them without intersecting p.

Are points B and C separated by path p in Figure 2?

Exercises 1-7a

(Class Discussion)

1. The figure at the right shows a simple closed path s and points L, M, N, O, P, Q, and R not on the path s.



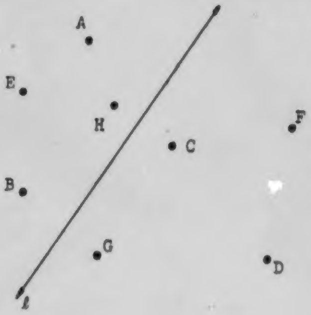
   (a) Name the points which are separated from point N by path s.

   (b) Name the points which are not separated from N by s.

   (c) Trace the drawing, and then draw paths on your tracing which connect point N to the points you named in (b).

   (d) Make another tracing of the drawing, and on it shade all the points on the page which are not separated from point N by path s. This set of points is called the interior of s.

2. The figure at the right shows a line ℓ and points A, B, C, D, E, F, G, and H not on ℓ.

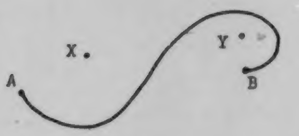


   (a) Which of the points named can be connected to point H by paths which do not intersect ℓ?

   (b) Which of the points named are separated from H by ℓ?

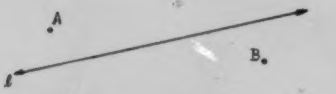
   (c) Imagine all the points in the plane which are separated from H by ℓ. Copy the figure above, and shade the part of the plane separated from H by ℓ.

3. Show why points X and Y are not separated by the simple open path connecting A and B.



4. Show why a segment does not separate any points in the plane in which it lies.

A line separates points in a plane just as a simple closed path does. Suppose line ℓ separates points A and B in some plane.



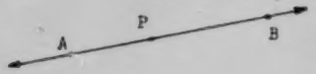
The line ℓ separates the plane into two sets of points:

All the points not on ℓ in the plane which are separated from point A by ℓ, and

All the points not on ℓ in the plane which are separated from point B by ℓ.

Each of these sets of points is called a half-plane. Together they form a pair of opposite half-planes. Line ℓ is the edge or boundary of the two half-planes. We sometimes name a half-plane by one of its points; for example, the "B half-plane" contains, among many others, point B. When there wouldn't be any confusion, we can call the B half-plane determined by line ℓ the "B-side of ℓ", and opposite half-planes determined by ℓ can be called "opposite sides of ℓ".

Just as a line separates a plane into two half-planes, a *point* separates a *line* into two *half-lines*.



The point P separates the line into two sets of points:

All the points except P on the line which are separated from point A by P, and

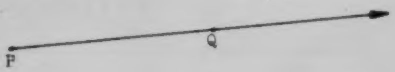
All the points except P on the line which are separated from point B by P.

Each of these sets is called a *half-line*. Together they form a pair of *opposite* half-lines. Point P is the *boundary* of the two half-lines.

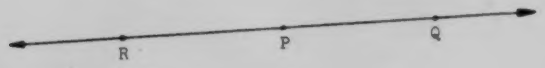
Point P is not contained in either of the half-lines it determines. If we join point P to one of its half-lines, the resulting set of points is called a *ray*.

Definition. *A ray is a half-line together with its boundary point. The boundary point is called the endpoint of the ray*.

If P is the endpoint of a ray and Q is another point of the ray, we name the ray "$\overrightarrow{PQ}$". Note that $\overrightarrow{PQ}$ is not the same as $\overrightarrow{QP}$. We now have a geometric meaning for the term "ray" which should match very well the picture we have of a "ray of light".

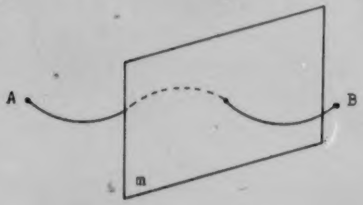


Two opposite half-lines, each joined with the common boundary point, form a pair of *opposite* *rays*. For example,



$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are opposite rays; point P is their common endpoint. Note that it is always true that if P is between Q and R, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are opposite rays, and $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ together form a line.

It should now come as no surprise that a plane separates space into two half-spaces. We will assume that points A and B in space are separated by plane m if each path from A to B in space intersects m.



In the figure, plane m separates space into two sets of points:

> All the points not on m in space which are separated from point A by m, and
>
> All the points not on m in space which are separated from point B by m.

Each of these sets is called a half-space. Plane m is the boundary of the two half-spaces.

Exercises 1-7b

1. We now know that

    (I) a simple closed path separates a plane
    (II) a point separates a line
    (III) a line separates a plane
    (IV) a plane separates space.

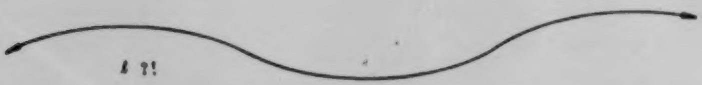
    Each of the following is a physical example of one of these separation ideas. For each, write the Roman Numeral of the separation idea which the example suggests to you.

    (a) the net on a tennis court
    (b) a shut-off valve on a pipeline
    (c) a moat around a castle
    (d) a dividing line down the middle of a road

(e) a station on a railroad line
(f) a window in your living room
(g) the plane of a movie screen
(h) the shore of a lake
(i) the Mason-Dixon line

2. Draw a line ℓ and mark points A and B on opposite sides of ℓ.

(a) Will every path (in the plane) from A to B intersect ℓ?
(b) Draw $\overline{AB}$. Does it intersect ℓ? (Can $\overline{AB}$ be thought of as a path?)
(c) Mark points C and D so that $\overline{AC}$ and $\overline{AD}$ intersect ℓ. On which side of ℓ must you mark C and D?
(d) Mark point E so that $\overline{AE}$ does not intersect ℓ. On which side of ℓ must you mark E?
(e) Using parts (b) and (c), state a simple test for showing that some points P and Q are on opposite sides of ℓ.
(f) Using part (d), state a simple test for showing that some points R and S are on the same side of ℓ.
(g) Would your tests in (e) and (f) still work if ℓ looked "wiggly"?



3. Not counting the lines themselves, into how many sets of points is a plane separated by two lines in the plane

(a) if the lines are parallel?
(b) if the lines intersect?

4. Tell whether each of the following statements is true or false. Draw a diagram supporting your answer.

(a) A segment contains many other segments.
(b) Each segment contains two rays.
(c) The words "ray" and "half-line" have the same meaning.
(d) A half-plane contains many rays.
(e) A ray contains many other rays.
(f) A half-plane contains many other half-planes.

5. In each case, identify the boundary between

(a) two half-lines.

(b) two half-planes.

(c) two half-spaces.

6. Tell whether each of the following is true or false.

(a) Space is separated by a half-plane.

(b) Space is separated by a line.

(c) Space is separated by a ray.

(d) Space is separated by a closed box.

(e) Space is separated by an open box.

(f) Space is separated by an inflated volley ball.

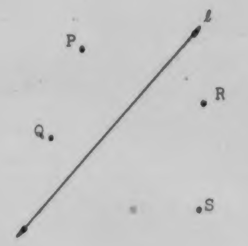
(g) Space is separated by a point.

7. Not counting the planes themselves, into how many sets of points is space separated by two planes,

(a) if the planes are parallel?

(b) if the planes intersect?

8. In the figure at the right, line ℓ and points P, Q, R, and S are in one plane. Are the following statements true or false?



(a) The R-side of ℓ is the same as the S-side of ℓ.

(b) The S-side of ℓ is the same as the P-side of ℓ.

(c) $\ell \cap \overline{PQ}$ is empty.

(d) $\ell \cap \overleftrightarrow{PQ}$ is empty.

(e) $\ell \cap \overrightarrow{PQ}$ is empty.

9. Draw a horizontal line. Label four points on it, A, B, C, and D in that order from left to right. Name two rays

(a) which are opposite.

(b) whose intersection contains exactly one point.

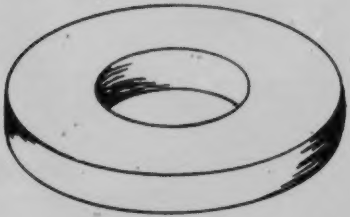
(c) whose intersection is empty.

(d) whose intersection is $\overline{CD}$.

BRAINBUSTER #1

The geometric idea which is a model for the surfaces of shapes like doughnuts, tire innertubes, or candy Life Savers is called a torus.

Find a simple closed path on a torus which does not separate any points on the surface of the torus.



BRAINBUSTER #2

A child wandered off from his parents in a park that was fenced, but had several gates. Guards at the gates reported he had passed through their gates as follows:

Gate #1 -- 3 times
Gate #2 -- 7 times
Gate #3 -- 5 times
Gate #4 -- 7 times

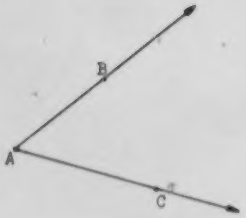
Where would you look for the child?

1-8. Angles

Angles have many applications in our modern civilization. An airplane pilot and a ship navigator use angles to chart their courses. An engineer builds a road so that the angle that the road makes with the horizontal plane is not too steep. A builder makes sure that the plane of a wall and the plane of the floor meet at right angles. In this section, we will review the concept of angle and some of the ideas that are related to it. However, we first need to recall the idea of the "union" of two sets.

Definition. The union of two sets of points is the set consisting of all the points belonging to either of the sets (or to both).

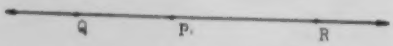
For example, the figure at the right represents the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ because it consists of all the points in $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Using the symbol "$\cup$" to stand for "union", we can write "$\overrightarrow{AB} \cup \overrightarrow{AC}$" to stand for the figure at the right, the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



Furthermore, the figure shown also represents what we mean by the word "angle". Therefore, we can say:

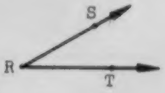
Definition. An angle is the union of two non-collinear rays having the same endpoint.

Why "non-collinear"? Remember that sets of points are collinear if there is a line which contains them. Therefore, if we allowed collinear rays to form an angle,

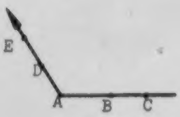


then the line formed by opposite rays $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ would have to be thought of as a kind of "angle". We will agree for now that no angle can be the same as a line, and hence, that any angle is made up of rays which are not on the same line, that is, "non-collinear" rays. (Perhaps the class discussion exercises will give you further reason for wanting the definition of angle to exclude collinear rays.)

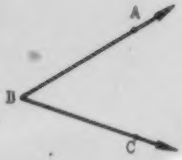
In the figure at the right, we have a representation of an angle. Point R, the common endpoint, is the vertex of the angle, and $\overrightarrow{RS}$ and $\overrightarrow{RT}$ are the sides of the angle. In order to be able to talk about angles we must agree on a method of naming them.



The angle illustrated at the right might be called ∠EAB, or ∠EAC, or ∠BAD, or ∠DAC. Note that the letter at the vertex is always written between the letters on the rays. In cases where no confusion is likely to result we name the angle by naming the vertex. In this case, we would refer to ∠A.



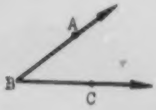
Do you think that every angle is contained in exactly one plane? Let us see why this must be so. Since points A, B, and C do not lie on the same line (they are not collinear) they must lie in exactly one plane by Property 4. Thus, we may talk about the plane of an angle.



Exercises 1-8a

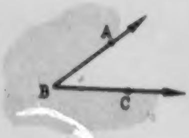
(Class Discussion)

1. Draw an angle ABC.

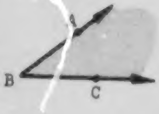


2. Does the angle separate the plane? If so, how can we describe the parts of the plane?

3. Shade the portion of the plane that appears outside the angle, as shown. This shaded portion is called the exterior of the angle.



4. Shade the portion of the plane that appears inside the angle, as shown. This shaded portion is called the interior of the angle.

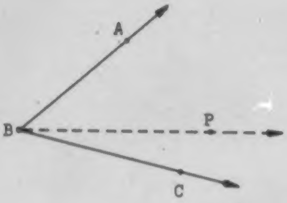


5. (a) Describe the A-side of $\overleftrightarrow{BC}$.
   (b) Draw a new $\angle ABC$ and shade the A-side of $\overleftrightarrow{BC}$.
6. (a) Describe the C-side of $\overleftrightarrow{AB}$.
   (b) On the figure you drew in part 5(b), shade in a different way the C-side of $\overleftrightarrow{AB}$.
7. Describe the doubly-shaded region in your drawing in 6(b).

These exercises suggest the following definition:

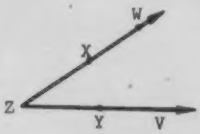
Definition. The interior of $\angle ABC$ is the intersection of the A-side of $\overleftrightarrow{BC}$ and the C-side of $\overleftrightarrow{AB}$.

If point P is in the interior of $\angle ABC$ we say that $\overrightarrow{BP}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Note that betweenness for rays is similar to but not identical to betweenness for points.



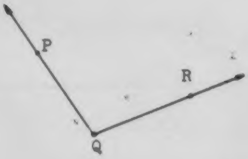
Exercises 1-8b

1. In the figure
   (a) name $\angle Z$ in two other ways.
   (b) name the vertex of $\angle Z$.
   (c) name the rays of $\angle Z$.



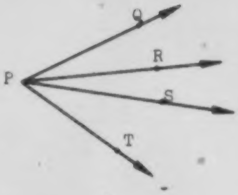
2. Label three points A, B, and C not all on the same line. Draw $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.
   (a) Shade the C-side of $\overleftrightarrow{AB}$.
   (b) Shade the A-side of $\overleftrightarrow{BC}$.
   (c) Describe the set that is doubly shaded.

3. For the figure at the right, write another name for
   (a) $\overrightarrow{QP} \cup \overrightarrow{QR}$
   (b) $\overrightarrow{QP} \cap \overrightarrow{QR}$



4. Tell whether each of the following is true or false.
   (a) The vertex of an angle is in the interior of the angle.
   (b) If two rays are not on the same straight line they form an angle.
   (c) An angle is the union of its interior and its exterior.
   (d) An angle separates its plane.
   (e) A point is in the interior of an angle if it is on a line which intersects the sides of the angle.

5. (a) Name three angles that have $\overrightarrow{PS}$ as a side.



(b) What is $\overrightarrow{PQ} \cup \overrightarrow{PS}$?

(c) What is the intersection of the interior of $\angle QPS$ and the interior of $\angle RPS$?

6. Suppose $\overrightarrow{OB}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OC}$, and $\overrightarrow{OX}$ is between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. Then

(a) $\overrightarrow{OB}$ is also between ___?___ and ___?___.

(b) $\overrightarrow{OX}$ is also between ___?___ and ___?___.

7. Draw $\angle ABC$.

(a) Draw a few rays between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Imagine all such rays drawn. How can you describe the figure formed?

8. Draw $\angle ABC$.

(a) In plane ABC draw a few rays with endpoint B that are not between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Imagine all such rays drawn. How can you describe the figure formed if $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are excluded?

Exercises 1-8c

(Class Discussion)

1. (a) Points X, Y, and Z are non-collinear.
Draw a figure representing the union of $\overline{XY}$, $\overline{YZ}$, and $\overline{XZ}$.

(b) Points L, M, and N are collinear.
Draw a figure representing the union of $\overline{LM}$, $\overline{MN}$, and $\overline{LN}$.

(c) The figure in part (a) is called a triangle, and we name it triangle XYZ or "$\triangle XYZ$". Is $\triangle XYZ$ a plane figure? How do you know?

(d) Using part (a), write a definition of a triangle beginning "Triangle XYZ is the union of ...".

(e) Is a triangle a simple closed path?

2. Mark 3 non-collinear points A, B, and C.
Draw $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overleftrightarrow{AC}$.

(a) Shade the A-side of $\overleftrightarrow{BC}$.

(b) Shade the B-side of $\overleftrightarrow{AC}$. Name the set of points which is doubly shaded.

(c) Shade the C-side of $\overleftrightarrow{AB}$. The set of points which is shaded three times is called the interior of $\triangle ABC$.

(d) Using parts (a), (b), and (c) and the idea of intersection of sets, write a definition for the interior of $\triangle ABC$, beginning "the interior of $\triangle ABC$ is the intersection of ...".

(e) Pick some point P in the interior of $\triangle ABC$. Pick another point Q which is not in the interior of $\triangle ABC$ and which is not on the triangle either. Does $\overline{PQ}$ intersect $\triangle ABC$? Always?

Definition. For any three non-collinear points A, B, and C, the union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called a triangle.

1-9. Locating Positions and Points

Below is shown a seating chart for a class in which the rows, and the seats in each row are numbered.

5	Kay	Mike	Nora	Eve	Carl
4	Fred	Pete	Gary	May	Nell
3	Ray	Ed	June	Myra	Paul
2	Ann	Emma	Bill	Don	Max
1	John	Mary	Jane	Jim	Sue
SEAT / ROW	1	2	3	4	5

Figure 3

From the chart we see that Ed sits in Row 2, Seat 3. This could be shortened to (R2,S3).

If we agree that the number for the row is named first and the seat number is named second then Ed's position is located by the pair of numbers (2,3). In this way a pair of numbers can be used to locate a position in a plane.

With the same agreement in naming the row and seat, Bill's position is located by the pair (3,2). Therefore, the pair (2,3) and the pair (3,2) locate different positions. Since the order of the pair of numbers is important in describing a position, we shall agree to the order, (row, seat) for this discussion.

Exercises 1-9a

(Class Discussion)

1. (a) Who sits at (4,5)?
   (b) Who sits at (5,4)?
   (c) What pair of numbers gives Pete's position?
   (d) What pair of numbers gives Don's position?

2. Write the set of all the number pairs which correspond to seats occupied by girls.

3. Consider the set of students:

   {Fred, Pete, Gary, May, Nell}.

   (a) What do their positions have in common?
   (b) Write the set of number pairs corresponding to these students.
   (c) What do these number pairs have in common?

4. Write the set of number pairs corresponding to the set of students:

   {Jane, Bill, June, Gary, Nora}.

   State what the number pairs have in common and what the positions of these students have in common.

5. (a) What is the intersection of the sets of students in Exercises 3 and 4?
   (b) What is the intersection of the sets of number pairs in Exercises 3 and 4?

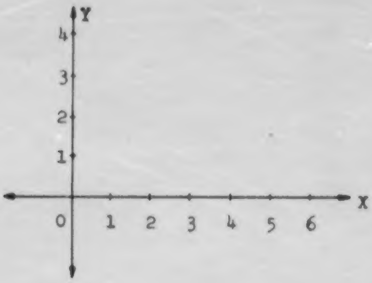
6. (a) Find the set of students having row number and seat number equal.
   (b) Find a geometric way of describing the positions of these students.

7. (a) Find the set of students having their seat number greater than their row number.
   (b) How can you express their positions geometrically?

The idea of using a pair of numbers to locate a position on a seating chart can be applied to locating points in a plane.

First, draw two lines, one horizontal and one vertical.

Notice that the four angles determined by these lines are all right angles. The point of intersection of the two lines is to be the zero-point (origin) on both lines. Using the same unit of distance on both lines, we can mark off points on each line at equal intervals.



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The horizontal line is called the X-axis and the vertical line is called the Y-axis. The plural of the word "axis" is "axes" (pronounced ack-seeze, with the accent on the ack). The point of intersection of the axes is called the origin.

Now let's see how a point such as S in Figure 4 below can be located.

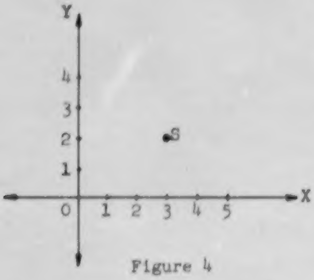


Figure 4

First, draw a vertical line ℓ_1 through S as in Figure 5.

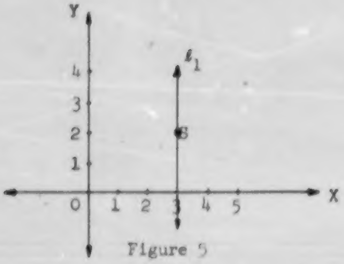


Figure 5

Look at the point where ℓ_1 crosses the X-axis. In this case, the vertical line crosses the X-axis at 3. Now look at Figure 6, where a horizontal line ℓ_2 has been drawn through S.

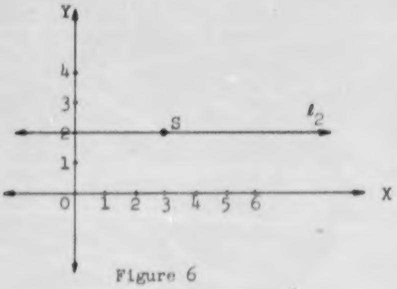


Figure 6

Here, the horizontal line crosses the Y-axis at 2. We can now locate S by the vertical line through 3 on the X-axis and by the horizontal line through 2 on the Y-axis. With the agreement that the number on the X-axis is to be named first, S is then located by the number pair (3,2). If we denote a number on the X-axis by the letter x and a number on the Y-axis by the letter y, we can write

$$(x,y) = (3,2).$$

Thus, we have a way of naming various points in the plane.

Drawing a point in the plane when the number pair is given for it is called plotting the point. To plot a point, we simply reverse the process described above. For example, if we were to plot (5,3), we draw the vertical line through 5 on the X-axis and the horizontal line through 3 on the Y-axis. Where these two lines intersect is the point (x,y) = (5,3).

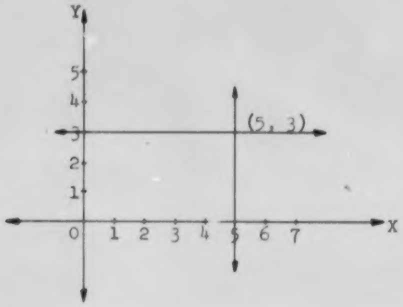
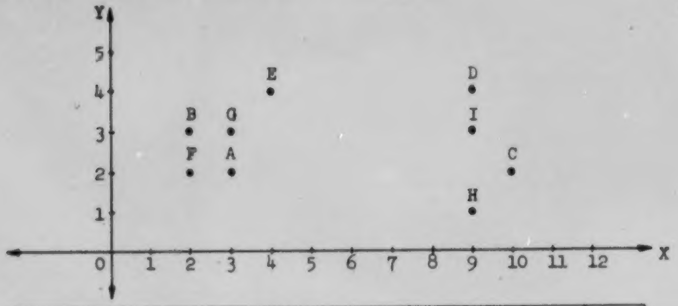


Figure 7

Exercises 1-9b

1. Using the picture below, fill in the table.



Point	A	B	C	D	E	F	G	H	I
x	?	?	?	?	?	?	?	?	?
y	?	?	?	?	?	?	?	?	?

2. Give the number pair for each of the points in Exercise 1.

Point	A	B	C	D	E	F	G	H	I
Number Pair	?	?	?	?	?	?	?	?	?

3. (a) What do points B, G, and I have in common?
   (b) What do the number pairs for B, G, and I have in common?

4. (a) What do points E, F, and G have in common?
   (b) Why might you say that the point (91,91) would have something in common with E, F, and G?
   (c) How might the point (91,91) be related geometrically to E, F, and G?

5. (a) What do points D, H, and I have in common?
   (b) If a point (x,59) has the same common property, what is the number x?

6. Plot the points located by the number pairs given below.

Point	P	Q	R	S	T	U
Number Pair	(1,4)	(6,1)	(4,4)	(0,0)	(2,5)	(0,5)

Point	V	W	X
Number Pair	(3,4)	(5,2)	(4,0)

7. (a) What do points Q, T, V, and W have in common?
   (b) If (1,y) has the same common property, what is y?

8. Give a number pair for a point between P and V that has the same common property shared by P, R, and V.

9. How can you tell by the number pair whether a point is on the X-axis?

Thus far, we have been able to locate points that can be named by pairs of whole numbers.

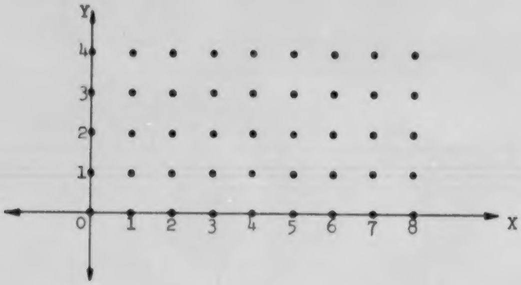


Figure 8

There are many more points which can be located in the part of the plane shown in Figure 8 by using such pairs of numbers as $(\frac{3}{2}, \frac{9}{13})$, $(\frac{17}{5}, \frac{22}{3})$ and so on. In this way more and more points of this part of the plane can be filled in until our picture begins to look like the following.

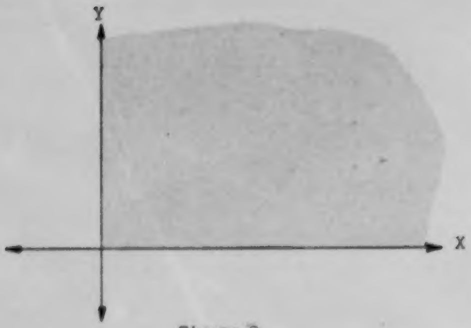


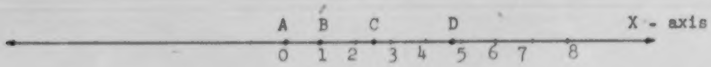
Figure 9

Notice that this still gives us only part of the plane. No point to the left of the Y-axis nor any point below the X-axis has been identified.

In the next sections we shall develop a way of locating and naming these other points.

1-10 Coordinates

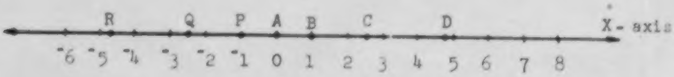
In constructing the horizontal number line called the X-axis, the point of intersection with the Y-axis (point A below) was assigned the number 0. A unit point (point B below) was selected and assigned the number 1. Then points at unit intervals were marked along the ray $\overrightarrow{AB}$ and assigned numbers as shown below.



You will see eventually that each point on ray $\overrightarrow{AB}$ can be assigned a number, as for example, the number for C is $2\frac{1}{2}$ or $\frac{5}{2}$ and for D is 4.8. The numbers assigned to the points on the line are called coordinates of the points. The coordinate of point B is 1 and the coordinate of D is 4.8 on the X-axis above. What is the coordinate of A? of C?

Now consider the opposite ray of ray $\overrightarrow{AB}$ on the X-axis. Mark a point (point P below) at a unit interval to the left of A. The coordinate of point P is $^{-}1$ (read "negative one") which is the "opposite" of the number 1.

The coordinate of the next interval point is $^{-}2$ (negative 2) which is the "opposite" of 2. This process is continued as shown below.



In this way numbers are assigned to each point on ray $\overrightarrow{AP}$, as for example, the coordinate of point Q is $^{-}2\frac{1}{2}$ or $-\frac{5}{2}$ which is the opposite of the coordinate of point C. What is the coordinate of point R if it is the opposite of the coordinate of point D?

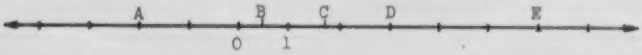
The numbers assigned to the points on half-line AP are called negative numbers, and the numbers assigned to points on half-line AB are called positive numbers. The number 0 is neither positive nor negative.

The Y-axis may be "coordinatized" in the same way as was just shown for the X-axis.

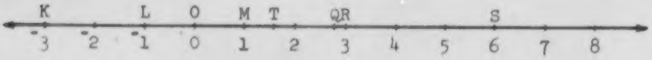
Exercises 1-10

1. Describe how positive and negative numbers may be assigned as coordinates of points on the Y-axis.

2. Referring to the diagram below, give the coordinates of each of the points E, A, B, and C, if C is located $\frac{2}{3}$ the distance from 1 to 2, and if B is midway between points with whole numbers for coordinates.



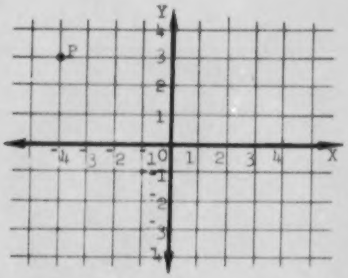
3. Give the letter name in the diagram below for the point whose coordinate is each of the following: $\frac{3}{2}$, $^{-}3$, 2.8, $^{-}1$, 0.



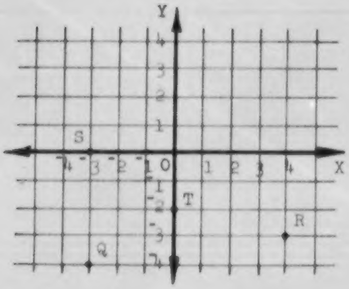
1-11. Coordinates in the Plane

Now that we have a way of assigning both positive and negative numbers (coordinates) to points on the X-axis and Y-axis, we can continue with locating points in various parts of the plane.

In the drawing below the horizontal and vertical lines are shown to help locate points. The vertical line through P crosses the X-axis at ¯4 and the horizontal line through P crosses the Y-axis at 3.



Then point P is located by the pair (¯4,3) where ¯4 is called the X-coordinate and 3 is called the Y-coordinate of point P. Notice that the X-coordinate of P is negative and the Y-coordinate of P is positive.

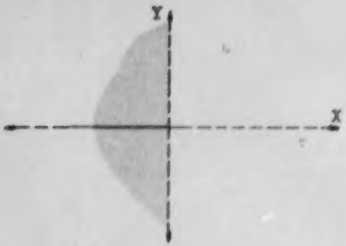


For point Q, you can tell at a glance that both coordinates are negative. Its coordinates are (¯3,¯4).

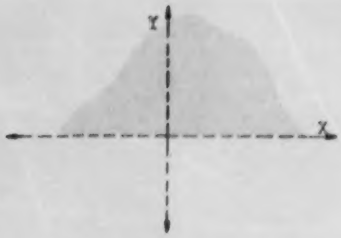
For point R, which coordinate is positive and which is negative? The coordinates for R are (4,¯3).

The Y-coordinate for S is 0. Why? Its coordinates are (¯3,0). Which coordinate of T is 0? The coordinates of T are (0,¯2).

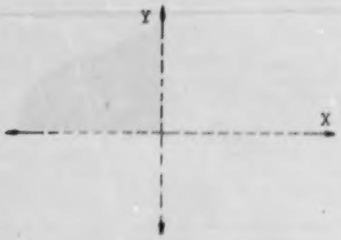
A plane in which each point is assigned a pair of coordinates and each pair of coordinates designates a point is called a coordinate plane. We can tell in which part of the coordinate plane a point lies by glancing at its coordinates. For example, let P be a point which has a negative X-coordinate and a positive Y-coordinate.



All points with negative X-coordinates lie in the half-plane to the left of the Y-axis, as shaded in the diagram.

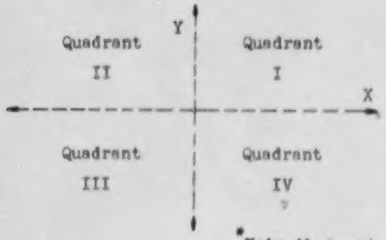


All points with positive Y-coordinates lie in the half-plane above the X-axis as shaded in the diagram.



All points with negative X-coordinates and positive Y-coordinates lie in the intersection of the two half-planes.

Therefore, P lies in the part of the coordinate plane which is shaded in the diagram to the left.

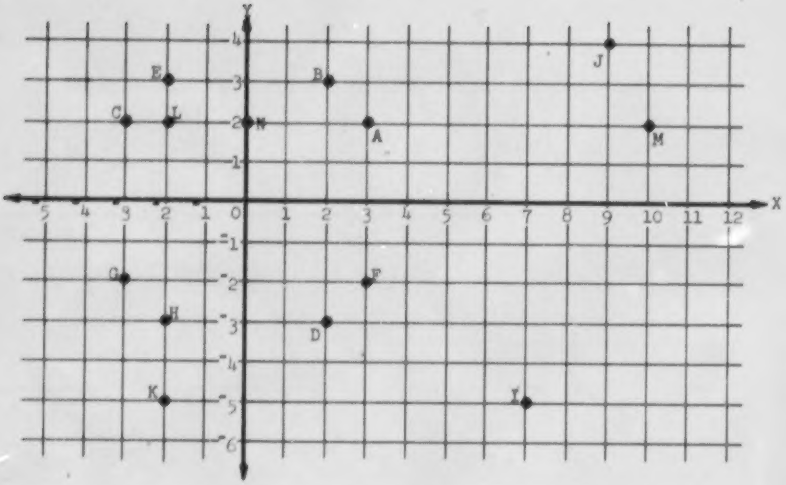


The four intersections of the half-planes for the X- and Y-axes are called quadrants. They are labeled with Roman numerals I, II, III, and IV as shown at the left.

*Note that points on the X- and Y-axes are in none of the quadrants.

Exercises 1-11

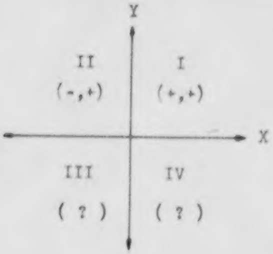
1.


Y
X
4
3
2
1
-1
-2
-3
-4
-5
-6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10
11
12
A
B
C
D
E
F
G
H
I
J
K
L
M
N

Using the diagram above, fill in the following table.

Point	A	B	C	D	E	F	G	H	I	J	K	L	M	N
Coordinates	?	?	?	?	?	?	?	?	?	?	?	?	?	?

2. For the diagram below, in Quadrant I, the symbol "(+,+)" means that the coordinates of the points are both positive.


Y
X
II
(-,+)
I
(+,+)
III
(?)
IV
(?)

In Quadrant II, the symbol "(-,+)" means that the points have negative X-coordinates and positive Y-coordinates.

Write the symbols for the coordinates of points in Quadrant III and Quadrant IV.

3. Plot the following points and draw line segments connecting A to B, to C, ..., and finally, from P to A. [The notation, A(0,9) means that A is a letter used to name the point whose coordinates are (0,9).]

A $(0,9)$, B $(5,5)$, C $(2,6)$, D $(6,2)$, E $(2,3)$.
F $(7,{}^-2)$, G $(1,{}^-2)$, H $(1,{}^-7)$, I $({}^-1,{}^-7)$, J $({}^-1,{}^-2)$,
K $({}^-7,{}^-2)$, L $({}^-2,3)$, M $({}^-6,2)$, N $({}^-2,6)$, P $({}^-5,5)$.

4. Do the same as for Exercise 3 for the points listed below.

A $(6,1)$, B $(8,4)$, C $(7,0)$, D $(10,{}^-3)$, E $(6,{}^-1)$,
F $({}^-3,{}^-5)$, G $({}^-9,{}^-4)$, H $({}^-6,{}^-2)$, I $({}^-9,{}^-2)$,
J $({}^-7,1)$, K $(0,4)$, L $({}^-1,2)$.

5. (a) Plot the following points: A $(7,{}^-3)$, B $({}^-2,10)$, C $({}^-2,{}^-6)$, D $(7,7)$, E $({}^-8,2)$.

(b) Draw the line segments: $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$.

6. Without plotting, give the quadrants of the points whose coordinates are given below.

(a) $(3,5)$
(b) $(1,{}^-4)$
(c) $({}^-4,{}^-4)$
(d) $({}^-3,{}^-1)$
(e) $(8,6)$
(f) $({}^-7,1)$
(g) $({}^-3,{}^-5)$
(h) $(0,2)$
(i) $(31,{}^-31)$
(j) $({}^-7,{}^-72)$
(k) $({}^-51,25)$
(l) $({}^-27,{}^-100)$

7. Plot all of the following points on one coordinate plane.

(a) A $(\frac{7}{4}, 2)$
(b) B $(-\frac{1}{2}, -\frac{1}{2})$
(c) C $(\frac{10}{3}, -\frac{4}{3})$
(d) D $(-\frac{5}{2}, 0)$
(e) E $({}^-6, \frac{11}{2})$
(f) F $(-\frac{9}{4}, -\frac{9}{2})$
(g) G $(0, \frac{13}{4})$
(h) H $(\frac{1}{2}, {}^-2)$
(i) I $({}^-6, -\frac{3}{2})$
(j) J $(0,0)$

8. Plot all the following points on one coordinate plane:

(a) A $(1.4, 5.3)$	(d) D $(^-3, 3.3)$
(b) B $(^-5.5, ^-3.5)$	(e) E $(0, ^-1.5)$
(c) C $(6, ^-2.7)$	

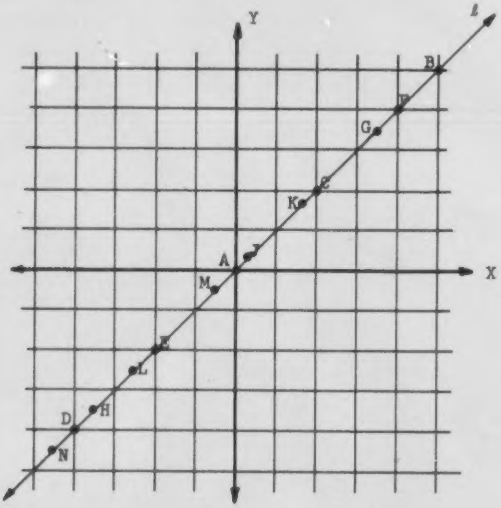
1-12. Graphs in the Plane

The study of geometry with the use of coordinates is called coordinate geometry. This branch of mathematics was started by the mathematician René Descartes in 1637. This invention was a great step forward in mathematics and made possible the discovery of calculus which followed shortly after. In this section, you will learn some very simple coordinate geometry.

Exercises 1-12a

(Class Discussion)

On the coordinate plane below, we have drawn the line ℓ through the point A $(0,0)$ and the point B $(5,5)$.



1. Find the coordinates of some other points on this line. Copy and complete the table below:

Point	A	B	C	D	E	F
X-coordinate	0	5	2	?	?	?
Y-coordinate	0	5	?	?	?	?

2. What do you notice about the coordinates of these points?

3. Now try some points whose coordinates are not integers (the set of integers consists of the number zero, the counting numbers, and the opposites of the counting numbers). Estimate the coordinates as closely as you can.

Point	G	H	J	K	L	M	N
X-coordinate	$3\frac{1}{2}$	?	?	?	?	?	?
Y-coordinate	$3\frac{1}{2}$	?	?	?	?	?	?

4. Does the same relation between the X- and Y-coordinates that you noticed before still seem to hold?

5. What do you conclude about the coordinates of points on this line?

You should have noticed that for all the points that you checked in the last example, the X- and Y-coordinates are equal. This relation holds true for all points on this line. Do you suppose that all points with X- and Y-coordinates equal will lie on this line? Try to find out. Plot these points on the above graph.

$$P\ (1,1),\quad Q(^-3,{}^-3),\quad R\ (4.3,\ 4.3),\quad S\ (^-2\tfrac{2}{3},{}^-2\tfrac{2}{3}).$$

Do these points seem to lie on the line ℓ? It is true that all points whose X- and Y-coordinates are equal lie on the line ℓ.

Here is another way of looking at what you have just developed. Let (x,y) be a point somewhere in the plane. If $y = x$, then this point lies on line ℓ. If $y \neq x$, then this point does not lie on ℓ. We can think of the line ℓ as the "picture" of the equation $y = x$. We say that $\underline{\ell \text{ is the graph of the equation } y = x}$.

Exercises 1-12b

1. Plot points $(^-1, {}^-1)$ and $(6,6)$, and draw line ℓ through them.
   (a) Is the point $(2\frac{1}{2}, 2\frac{1}{2})$ on ℓ?
   (b) Is the point $(^-100, {}^-100)$ on ℓ?
   (c) Is the point $(7, 7\frac{1}{2})$ on ℓ?
   (d) Is the point $(^-5, 5)$ on ℓ?
   (e) If (a,b) is on ℓ, what must be true about the numbers a and b?
   (f) If (c,d) is not on ℓ, what must be true about the numbers c and d?
   (g) If x is any number, is the point (x,x) on ℓ?
   (h) Can you find a number n for which the point $(n, n+1)$ is on ℓ?

2. Think of points whose coordinates are $(5,y)$.
   (a) Plot $(5,y)$ where y is 1.
   (b) Plot $(5,y)$ where y is $^-2$.
   (c) Plot $(5,y)$ where y is $\frac{5}{3}$.
   (d) Plot $(5,y)$ where y is $^-4$.
   (e) Plot $(5,y)$ where y is $4\frac{1}{2}$.
   (f) Describe where these points lie.
   (g) The graph of all points whose coordinates are $(5,y)$ is ________.

3. Think of points whose coordinates are $(x,4)$.
   (a) Plot $(x,4)$ where x is 2.
   (b) Plot $(x,4)$ where x is $3\frac{1}{3}$.
   (c) Plot $(x,4)$ where x is $^-3$.
   (d) Plot $(x,4)$ where x is 0.
   (e) Plot $(x,4)$ where x is $^-5$.
   (f) Describe where these points lie.
   (g) The graph of all points whose coordinates are $(x, 4)$ is ________.

4. If the lines for Exercises 1, 2, and 3 were drawn on the same coordinate plane, what would be the coordinates of the point of intersection for the lines formed in

(a) Exercises 1 and 2?

(b) Exercises 1 and 3?

(c) Exercises 2 and 3?

5. Draw the line through points $G(^{-}5,5)$ and $H(5,^{-}5)$.

(a) Point J has an X-coordinate of 2, and J is on $\overleftrightarrow{GH}$. What is the Y-coordinate of J?

(b) Point K on $\overleftrightarrow{GH}$ has a Y-coordinate of $^{-}3$. What is the X-coordinate of K?

(c) Find the missing coordinates for points L, M, N and P which lie on $\overleftrightarrow{GH}$.

Point	L	M	N	P
X-Coordinate	$^{-}5$	?	117	$^{-}117$
Y-Coordinate		$^{-}5$	?	?

6. Pick two points A and B in Quadrant I for which the X-coordinate is twice the Y-coordinate. Plot those two points, and draw $\overleftrightarrow{AB}$.

(a) Is the point (3,6) on $\overleftrightarrow{AB}$?

(b) Is the point (6,3) on $\overleftrightarrow{AB}$?

(c) Is the point (20,10) on $\overleftrightarrow{AB}$?

(d) Is the point (2n,n) on $\overleftrightarrow{AB}$ for any positive number n?

(e) Is the point $(^{-}6,^{-}3)$ on $\overleftrightarrow{AB}$?

(f) Is the point $(^{-}100,^{-}50)$ on $\overleftrightarrow{AB}$?

(g) Is the point $(^{-}24{,}690\ ,\ ^{-}12{,}345)$ on $\overleftrightarrow{AB}$?

1-13. Summary

Section 1-1.

The physical objects around us suggest the ideas regarding shape and size which are studied in geometry.

We think of points as definite locations and space as the set of all points. Geometric figures are sets of points and therefore are subsets of space.

Section 1-2.

A line is a set of points suggested by a stretched string. Many lines can pass through a single point.

Property 1. Through any two different points in space there is exactly one line.

Lines are named by identifying two of its points.

If two lines intersect, they intersect in exactly one point called the point of intersection.

Section 1-3.

A plane is a set of points suggested by a flat surface extending indefinitely in all directions.

Property 2. If two points of a line lie in a plane then all of the line lies in the plane.

Property 3. For any two points in space, there are many planes which contain them.

Property 4. Through any three points, not all on the same line, there is exactly one plane.

A plane is named by identifying any three of its points not all on a line.

Section 1-4.

When two lines intersect, their intersection consists of exactly one point.

When the intersection of two lines is empty,

(a) they are called parallel if they lie in the same plane.
(b) they are called skew if they do not lie in the same plane.

Section 1-5.

The intersection of a line and a plane may

(a) be empty, in which case, they are parallel,
(b) consist of a single point,
(c) or be the line itself.

Property 5. If the intersection of two different planes is not empty then the intersection is a line.

When we say that two planes are parallel we mean that their intersection is the empty set.

Section 1-6.

When we say that point P is between points A and B, we mean

(a) A, B, and P are collinear, and
(b) on their line the points are in the order A, P, B or B, P, A.

Definition. The segment $\overline{AB}$ is the set of points consisting of the two points A and B, and all points between A and B.

Section 1-7.

A path in a plane is a set of points which can be represented by making a tracing with a pencil on a sheet of paper without lifting the pencil.

A simple closed path is a path in a plane which starts at a point and returns to that point without crossing itself.

Two points are separated by a path p in a plane if the two points are not on path p and if every path connecting the two points intersects path p.

A simple closed path separates the plane into two parts called the interior and exterior.

A line separates the plane into a pair of opposite half-planes or sides.

A point on a line separates the line into a pair of opposite half-lines.

A plane separates space into two half-spaces.

A ray is a half-line together with its boundary point. The boundary point is called the endpoint of the ray.

Two opposite half-lines, each joined with the common boundary point, form a pair of opposite rays.

Section 1-8.

An angle is the union of two non-collinear rays having the same endpoint.

The interior of $\angle ABC$ is the intersection of the A-side of $\overleftrightarrow{BC}$ and the C-side of $\overleftrightarrow{AB}$.

If point P is in the interior of $\angle ABC$, we say that $\overrightarrow{BP}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

Definition. For any three non-collinear points A, B, and C, the union of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called a triangle.

Section 1-9.

A point may be located in a plane by a pair of numbers (x,y) using an X-axis and a Y-axis. The order of the pair of numbers is important. The pair $(2,3)$ and the pair $(3,2)$ locate different points.

Section 1-10.

The X-axis is separated into two opposite half-lines by the origin, the point of intersection with the Y-axis. Positive numbers are assigned to all the points of the half-line to the right. Negative numbers are assigned to all points on the opposite half-line to the left as opposites of the positive numbers. Positive and negative numbers are assigned to points on the Y-axis in a similar manner.

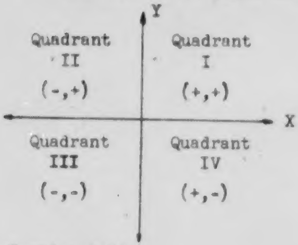
The numbers assigned to the axes are called coordinates.

Section 1-11.

With the assigning of negative numbers to the axes, points may be located in any part of the plane by pairs of numbers (x,y). The number x is called the X-coordinate and the number y is called the Y-coordinate of a point.

The coordinate plane is a plane in which each point is assigned a pair of coordinates and each pair of coordinates designates a point.

The coordinate plane is separated into four quadrants as shown at the left. In each quadrant the X- and Y-coordinates of the points are indicated as positive $(+)$ or negative $(-)$.



Section 1-12.

Points whose X- and Y-coordinates are equal lie on a line ℓ. Then for any point $P(x,y)$, if $y = x$, P is a point on ℓ, and ℓ is the set of all such points. We say that ℓ is the graph of the equation $y = x$.

In this way various lines become graphs consisting of points having X- and Y-coordinates related in certain ways.

Teacher's Commentary

Chapter 1

STRUCTURING SPACE

1-1. Introduction

Geometry is approached here as a subject that is based upon or modeled from our experience with physical space. The students by now have gained a substantial amount of experience and knowledge about the shape, form, and size of various physical objects in their surroundings. Reference should be made continuously to this background in developing geometric ideas.

The basic concepts of geometry -- point, line, plane, space -- do not actually appear in the physical world but are ideas that are suggested by various physical objects. This is the theme carried throughout as other geometric figures are formed. Space is idealized as the set of all points and the various figures are formed as subsets of points. The way in which the figures are formed determines the structure of our space.

The structuring of space is presented as a dynamic process of formulating the properties or characteristics we think our space should have. Each property is checked for consistency with our knowledge of physical space. These properties provide a basis for systematically organizing our knowledge and for focusing on the really fundamental elements of our structure.

In a coordinate system, points are assigned numbers in a systematic way. Consequently, we can represent geometric ideas numerically and study the properties of sets of points (figures) by means of numbers. Later we shall study the properties of numbers by means of our model of space. Furthermore, this sets the stage for the study of the important idea of function in the next chapter.

Exercises 1-1 (Class Discussion)

1. There are many possible completions, including:
   ... ideas based on the size, shape, and location of physical objects;
   ... models resulting from experiences with physical objects;
   ... relations among points, lines, and planes, in space; etc.
2. Geometry deals with ideas which are suggested by physical objects.
3. A knot on a thread,
   the tip of a pencil,
   the spot where the corners of 4 floor tiles meet,
   ... etc.

4. Very definite location, especially a model on chart from position in the physical world.

5. It could not have size, because if it did the location it represented would be indefinite.

6. ... what it is made of
... color
... texture
... weight
etc.

1-2. Lines and Points

The fundamental relationships between points and lines are studied in this section. We accept as a basic characteristic of our structure of space that for any two points in space there is one and only one line that passes through them.

The acceptance of this property leads to a rather surprising result. It can be argued from Property 1, as is done informally in the class discussion exercises, that two lines cannot intersect in more than one point. This is the first encounter for the student of an argument or proof presented very informally. It is an indirect proof, but the technicalities need not be discussed. Most students will see the reasoning that if the assumption of these being two points in the intersection of two lines gets us in trouble (leads to a contradiction) then we should discard the assumption.

Exercises 1-2a (Class Discussion)

1. (a) The intersection is thick and fuzzy. It simply looks like there is more than one point there.

   (b) Since A and B are contained in the intersection of ℓ and m, then

   ℓ contains A and B

   and m contains A and B

   so both ℓ and m contain, or pass through, points A and B.

(c) The assumption that there are two points in the intersection leads to the result that two lines may pass through the same two points.

Property 1, however, states that only one line may pass through two points.

These two statements cannot both be true so one of them is false.

If we agree that Property 1 is true, we must discard the assumption that there are two points in the intersection and agree that ℓ and m intersect in just one point.

If there were more than two points in the intersection, things would even be worse;
Property 1 would still be contradicted.

Exercises 1-2b

1. Answers will vary. Possibilities include:
the line of sight from your eye to a star,
a thin laser beam,
the intersection of a wall and floor,
etc.

2. (a) and (b): 12 names: $\overleftrightarrow{AB}$, $\overleftrightarrow{BA}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{DA}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CB}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{DB}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{DC}$.

3. (a) 3
   (b) There are many ways to name $\overleftrightarrow{AE}$, $\overleftrightarrow{KF}$, and $\overleftrightarrow{HD}$.
   (c) $\overleftrightarrow{AE}$ and $\overleftrightarrow{KF}$ intersect in point B.
   $\overleftrightarrow{KF}$ and $\overleftrightarrow{HD}$ intersect in point G.
   $\overleftrightarrow{AE}$ and $\overleftrightarrow{HD}$ intersect in point D.

4. (a) Student activity
   (b) $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{CD}$.
   (c) 6

5. 10

6. (a) B
   (b) D
   (c) E

7. (a) m and n
   (b) ℓ and m
   (c) p and n

1-3. Planes

In this section some very important structuring takes place. Each of the three properties stated here needs some demonstration to show that they are fundamental to our experience in physical space.

The essential message of Property 2 is that if a plane contains two points of a line it must contain all the points of the line. This means that the line and plane are then irrevocably committed to each other and will never part company. This ties together our ideas of the straightness of lines and the flatness of planes.

The demonstration with three finger tips and cardboard is helpful for developing Properties 3 and 4. Show that there are many positions for the cardboard fixed to one finger tip and to two finger tips, but that there is only one position for the cardboard when fixed to three finger tips not all on the same line. Since many planes may contain any two points it follows that many planes may contain the line determined by the two points.

The exercises offer an opportunity for students to give reasons or provid an informal argument. This will help the students to become aware that these properties are quite fundamental and lead to important results. Also, the students may begin to get an inkling of what is a structure of ideas. Note, particularly, Exercises 11 and 12 in which students are asked to show that a plane is determined by a line and a point not on the line, and is also determined by two intersecting lines.

It is hoped that students will become tired of saying, "not all on the same line". They will be ready for the terms "collinear" and "non-collinear" which are introduced later in Section 1-6.

Exercises 1-3

1. The ends of the 3 legs (of the stool or tripod) can be thought of as 3 points, and Property 4 states there is always a plane containing them. Therefore, the 3 legs can always be made to rest flat on the floor.

2. (a) yes
   (b) yes
   (c) no
   (d) no

3. (a) yes
   (b) yes
   (c) yes
   (d) no

4. 4 planes: ABC, ABD, BCD, ACD.

5. (a) $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are in plane ABC by Property 2.

   (b) Part (a) shows R and S are in plane ABC, and Property 2 says that as a result $\overleftrightarrow{RS}$ is in plane ABC.

6. (a) Points X, Y, and Z lie on the same line.
   (b) Points X, Y, and Z do not lie on the same line.

7. (a) an infinite number
   (b) exactly one

8. (a) an infinite number
   (b) an infinite number
   (c) exactly one

9. Many planes may contain the two points A, B. (Property 3)
   Each of these planes contains the line $\overleftrightarrow{AB}$. (Property 2)
   Therefore, many planes contain the line $\overleftrightarrow{AB}$.

10. If the surface were a plane, then the line $\overleftrightarrow{AB}$ would be wholly contained in it (Property 2). Since $\overleftrightarrow{AB}$ intersects it in just 2 points, the surface could not be a plane.

11. The 3 points A, B, and C lie in exactly one plane (Property 4) and $\overleftrightarrow{AB}$ lies in it also (Property 2), so $\overleftrightarrow{AB}$ and C lie in exactly one plane. The plane could be named plane ABC.

12. By Exercise 10, $\overleftrightarrow{AB}$ and C lie in exactly one plane, and by Property 2, $\overleftrightarrow{AC}$ lies in it also. Therefore, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ lie in exactly one plane. It could be named plane ABC.

13. (a) D
    (b) Answers may be any combination of 3 letters from {C,D,E,F}.
    (c) Answers may be any combination of 3 letters from {A,B,G,H}.
    (d) Answers may be any combination of 3 letters from {A,B,C,D}.
    (e) Answers may be any combination of 3 letters from {E,F,G,H}.

1-4. Intersections

Review for students the use of braces, { }, for set notation. The definition for intersection is given, and if necessary, you should provide other examples for finding intersections of sets.

Example: Let $A = \{3,7,11,15\}$ and $B = \{4,7,10,12\}$, then $A \cap B = \{7\}$.

It is well to introduce both symbols for the empty set "$\emptyset$" and "{ }", because they are both encountered in mathematics textbooks.

Point out that the intersection of a pair of sets is always a set. Also, point out that we also use "intersect" as a verb. Two lines may intersect or they may not intersect.

Parallel lines and skew lines are introduced here since there are two possibilities for lines which do not intersect.

Note especially Exercise 6 which summarizes the possible relative positions for two lines.

Do not insist upon rigid conformity with respect to always using braces when representing the results of forming the union or intersection of sets. It is the concept we're after here not the notation. Hopefully, the students will acquire the proper notation by observing the teacher and the text.

Exercises 1-4

1. (a) {4,10}
   (b) {John,Frank,Alice}
   (c) (Answers will vary.)
   (d) {July 4th}
   (e) $\emptyset$

2. (a) {B}
   (b) {D}
   (c) $\emptyset$
   (d) {C}

3. (a) Any 3 from $\overleftrightarrow{EF}$, $\overleftrightarrow{EG}$, $\overleftrightarrow{DH}$, and $\overleftrightarrow{CD}$.
   (b) $\overleftrightarrow{AB}$, $\overleftrightarrow{FH}$, and $\overleftrightarrow{ED}$.

4. (a) {D}
   (b) $\emptyset$
   (c) {A}
   (d) {H}

5. Not always; if they do not lie in the same plane, they are skew lines, not parallel lines.

6. (a) intersecting lines
   (b) parallel lines
   (c) skew lines
   (d) No
   (e)

	Intersect	Do not intersect
Lie in same plane	(a) intersecting lines	(b) parallel lines
Do not lie in same plane	(d) "No" (impossible)	(c) skew lines

1-5. Intersections of Lines and Planes

This section is a continuation of how lines and planes may be related to each other in terms of their possible intersections.

A line and a plane may have the following possible intersections:

(1) the empty set,
(2) the set consisting of a single point,
(3) the line itself.

In the case where the intersection of a line and a plane is the line itself, some additional explanation may be necessary. Note that in this case the line lies in the plane, or in other words, is a subset or portion of the plane. Each point of the line then is common to both the line and the plane. In general, whenever A is a subset of B then $A \cap B = A$.

Example: Let $A = \{1,3,4\}$ and $B = \{1,2,3,4,5\}$. Clearly A is a subset of B. Then $A \cap B = A$.

The class discussion exercises present a very informal argument that if two planes intersect they must intersect in a line.

Two planes then may have the following possible intersections:

(1) the empty set,
(2) a line.

The exercises provide opportunity for the students to explore the various possibilities about how lines and planes are related.

Exercises 1-5a (Class Discussion)

1. No; since the planes do not stop at the edges of the sheets of paper, the intersection of the planes will contain more than just the single point common to the sheets of paper.

2. (a) A and B lie in both planes. Property 2 says that, as a result $\overleftrightarrow{AB}$ must lie in both planes.

   (b) Property 4 and especially Exercise 10 in (1-3) show that $\overleftrightarrow{AB}$ and any other point not on $\overleftrightarrow{AB}$ may lie in exactly one plane. Therefore, they could not be in the intersection of (or be common to) two different planes.

Exercises 1-5b

1. (Answers will vary.)

2. (a) $\{M\}$
   (b) $\{D\}$
   (c) $\{B\}$
   (d) $\emptyset$
   (e) $\overleftrightarrow{AB}$
   (f) $\emptyset$

3. (a) Any 3 from among planes ABE, CDF, ABCD, BCFE, and EFDA. (Each of the last 3 may be named by any combination of 3 of the four letters mentioned here.)

   (b) $\overleftrightarrow{EF}$, $\overleftrightarrow{DF}$, and $\overleftrightarrow{CF}$

   (c) $\overleftrightarrow{AD}$ and $\overleftrightarrow{EF}$

   (d) plane ABE and plane CDF

   (e) plane ABCD and plane CDF

   (f) 3 possible answers: $\overleftrightarrow{AE}$ and plane BCEF, $\overleftrightarrow{BE}$ and plane ADEF, or $\overleftrightarrow{FE}$ and plane ABE.

   (g) plane BCEF and either BC or EF.

4. (a) $\overleftrightarrow{CF}$
   (b) $\{B\}$
   (c) $\{G\}$
   (d) $\{H\}$
   (e) $\overleftrightarrow{CG}$
   (f) $\emptyset$
   (g) $\overleftrightarrow{EC}$

5. (a) Planes EFGH and ABGH. (Some students may see that if points C, D, G, and H are in the same plane, then that plane CDGH could be one of the pair of planes which intersect in $\overleftrightarrow{HG}$.)

(b) Planes EFGH and BCFG. (As in part (a), plane ADFG could be included.)

(c) Any two from among planes ACE, BHCE, and GCE.

(d) Any two from among planes DCGH, ECBH, and ACFH. (The fact that D, C, G, and H all lie in the same plane may prompt discussion which is best handled by referring to a physical model of the box.)

6. The line ℓ may lie in the plane m: $\ell \cap m = \ell$.

The line ℓ may intersect the plane m in one point P:

$$\ell \cap m = \{P\}.$$

The line ℓ may be parallel to plane m: $\ell \cap m = \emptyset$.

7. The two planes may intersect: their intersection is a line.

The two planes may be parallel: their intersection is $\emptyset$.

1-6. Betweenness and Segments

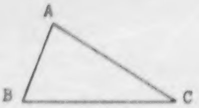
Betweenness for points has a very special meaning. The word "between" is used only for three points that lie on a line. The term "collinear" is introduced at this time and is used in the definition of betweenness.

The concept of betweenness is needed to define a segment and its use in the definition results in a segment being a subset of a line. Since a segment is uniquely determined by its endpoints, a segment is named by its endpoints.

Exercises 1-6a

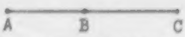
1. (a) no
   (b) yes
   (c) yes
   (d) yes
   (e) $\overline{TK}$, $\overline{TD}$, $\overline{KD}$
   (f) no
   (g) no
   (h) {M,B,T}, {M,C,K}, {M,R,D}, {B,C,R}, {T,K,D}

2. (a)



(b) A segment.

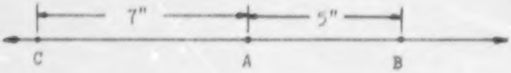
For example,



$\overline{AC}$

3. The word "between" in the definition of a segment requires that the points of a segment lie on a line.

4.



A is between C and B.

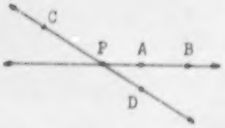
5. (a) $\overline{ZX}$, $\overline{ZY}$, $\overline{ZV}$, $\overline{ZW}$

(b) 3 are shown

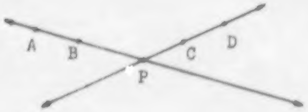
(c) $\overline{XY}$, $\overline{XV}$, $\overline{YV}$, and $\overline{ZY}$

(d) $\overline{WZ} \cap \overline{WV} = \{W\}$

6. Answers will vary. Here are two drawings.

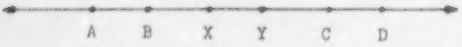


$\overline{AB} \cap \overline{CD} = \emptyset$



$\overleftrightarrow{AB} \cap \overleftrightarrow{CD} = \{P\}$

7. (a)



$\overline{AY}$, $\overline{AC}$, $\overline{AD}$, $\overline{BY}$, $\overline{BC}$, $\overline{BD}$ all contain points X and Y.

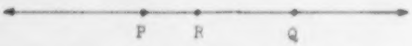
(b) infinitely many

(c) exactly one

(d) exactly one

(e) No. See 7(a) above.

BRAINBUSTER:



PQ > QR

QR > PR

R must be between P and Q

1-7. Separation

The important idea of separation is introduced through the intuitive notion that two points may be separated by certain kinds of paths.

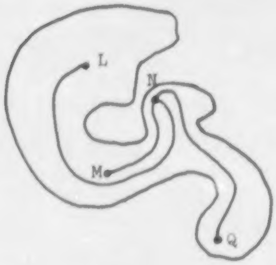
First, the concept of paths in a plane is developed. Then these are classified so that the meaning of a simple closed path is clear. A simple closed path separates two points that do not lie on it if every path connecting the two points intersects the simple closed path.

In the class discussion this idea is pursued further until the set of points called the interior of the simple closed path is identified. If you wish, you may continue and identify the set of points called the exterior. It is important that students see why some paths do not separate certain points.

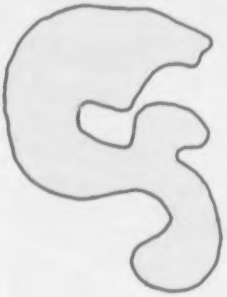
After the class discussion exercises the students are ready to examine the separation properties of points, lines and planes. In each case there is a boundary, a point, a line, or a plane, and two parts, called appropriately half-lines, half-planes, and half-spaces. With half-lines introduced, a ray can be identified as a half-line together with its boundary point. The concepts of opposite rays, opposite half-lines, and opposite half-planes or sides of a line are important because they will be used quite frequently.

Exercises 1-7a (Class Discussion)

1. (a) O, P, R
   (b) L, M, Q
   (c)



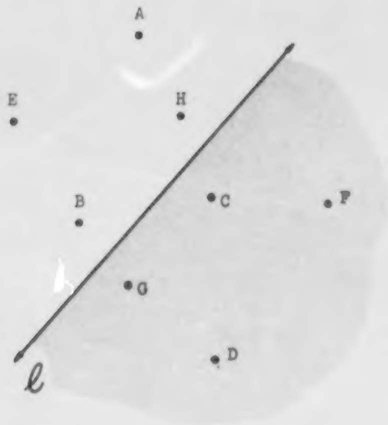
(d)



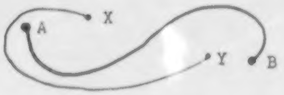
2. (a) A, E, B

(b) C, D, F, G

(c)



3. X and Y are not separated because there is a path from X to Y which does not intersect the simple open path connecting A and B.

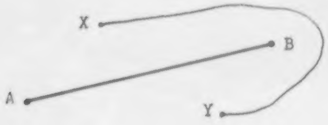


The path from X to Y can "go around" the path connecting A and B.

4. Let students draw an example.

If X and Y are any two points in the plane not on a segment $\overline{AB}$, X and Y can be connected by a path that does not intersect $\overline{AB}$. Therefore X and Y are not separated by $\overline{AB}$.

The path from X to Y can "go around" the segment.



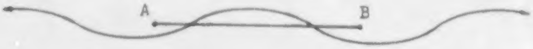
Exercises 1-7b

1. (a) IV (b) II (c) I (d) III (e) II (f) IV
(g) IV (h) I (i) III

2. (a) Yes
(b) Yes ($\overline{AB}$ can be thought of as a path.)
(c) C and D must be marked on the opposite side of A.
(d) E must lie on the same side as A.
(e) Two points P and Q not on a line ℓ lie on opposite sides of ℓ if and only if $\overline{PQ}$ intersects ℓ.
(f) Two points R and S not on a line ℓ lie on the same side of ℓ if and only if $\overline{RS}$ does not intersect ℓ.
(g) The "segment test" in (e) works for the "wiggly" path to test for points on opposite sides.

The "segment test" in (f) for points on the [illegible] side does not necessarily work for "wiggly" paths.

Example: A and B are on same side, yet $\overline{AB}$ intersects path.



Segment test works only for lines.

3. (a) three (b) four

4. (a) true (b) false (c) false
(d) true (e) true (f) true

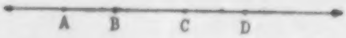
5. (a) a point (b) a line (c) a plane

6. (a) false (b) false (c) false
(d) true (e) false (f) true
(g) false

7. (a) three (b) four

8. (a) true (b) false (c) true
(d) false (e) false

9.



(a) $\overrightarrow{BA}$ and $\overrightarrow{BC}$ $\overrightarrow{CB}$ and $\overrightarrow{CD}$

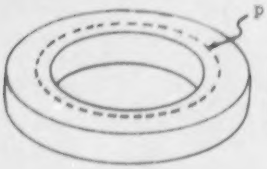
(b) $\overrightarrow{BA} \cap \overrightarrow{BC} = \{B\}$ $\overrightarrow{CB} \cap \overrightarrow{CD} = \{C\}$

(c) $\overrightarrow{BA} \cap \overrightarrow{CD} = \emptyset$

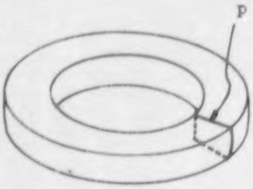
(d) $\overrightarrow{CD} \cap \overrightarrow{DC} = \overline{CD}$

BRAINBUSTER #1

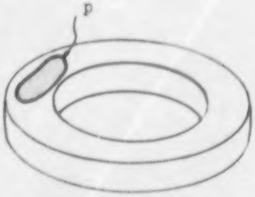
See if students can find two answers.



path p does not
separate surface
of a torus



path p does not
separate surface
of a torus



the boundary of the
patch does separate
the surface of a torus

BRAINBUSTER #2

3 + 7 + 5 + 7 = 22 (even number)

Since the child starts from inside the park and his path intersects the park boundary an even number of times, you would look for him inside the park.

1-8. Angles

To develop the definition of an angle it is necessary to review the concept of union of sets. "A ∪ B" means "the union of sets A and B", and the union consists of all the members of both sets. You might provide some examples.

Let $A = \{1,4,7,9\}$ and $B = \{2,4,6,7\}$.

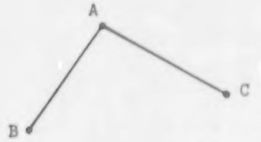
Then $A \cup B = \{1,2,4,6,7,9\}$.

The two sets in effect are "joined" to form the union.

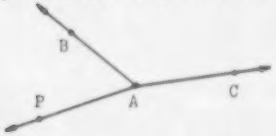
An angle is now defined as the union of two non-collinear rays having a common endpoint. Since an angle is determined by three non-collinear points with one of them designated as the vertex, we can use the three points to name the angle. The vertex is always named in the middle. If an angle is named by three points, we can name the rays that form the angle. For example, $\angle BDC$ is formed by the rays $\overrightarrow{DB}$ and $\overrightarrow{DC}$. $\angle BDC = \overrightarrow{DB} \cup \overrightarrow{DC}$. Since the three non-collinear points which determine an angle also determine a plane, we can speak of the plane of an angle.

The requirement that the rays forming an angle be non-collinear eliminates "straight angles" and "zero angles". The reason for agreeing that an angle cannot be the same as a line or a ray becomes apparent in the class discussion exercises where interiors and exteriors of angles are discussed.

Two non-collinear segments with a common endpoint may suggest an angle although the rays are not shown. For example, the two segments $\overline{AB}$ and $\overline{AC}$ shown at left suggests the angle $\angle BAC$ even though the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not shown. This allows us to tell later about angles of triangles and angles of various polygons.



Betweenness for rays is defined. If P is a point in the interior of $\angle BAC$, then $\overrightarrow{AP}$ is between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. It might be well to show an example where the ray $\overrightarrow{AP}$ is not between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as is shown at the left.



In the final class discussion exercises the definition of a triangle is developed and the interior of a triangle is shown to be the intersection of three half-planes.

Exercises 1-8a (Class Discussion)

1. Show drawing.

2. Yes. Two parts of the plane can be identified so that every path from any point in one part to any point in the other part intersects the angle.

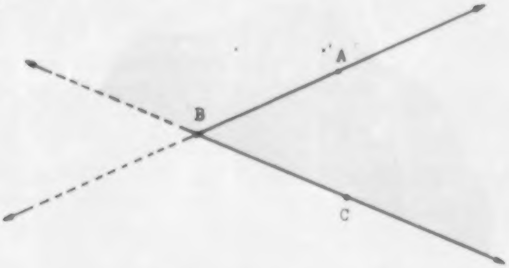
   The parts can be described as the inside part and the outside part.

3. Done on drawing.

4. Done on drawing.

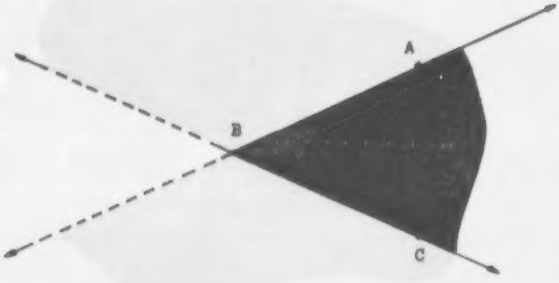
5. (a) The A-side of $\overleftrightarrow{BC}$ is the half-plane with boundary $\overleftrightarrow{BC}$ which contains point A.

   (b)



6. (a) The C-side of $\overleftrightarrow{AB}$ is the half-plane with boundary $\overleftrightarrow{AB}$ which contains point C.

   (b)



7. The doubly shaded region represents the intersection of the two half-planes.

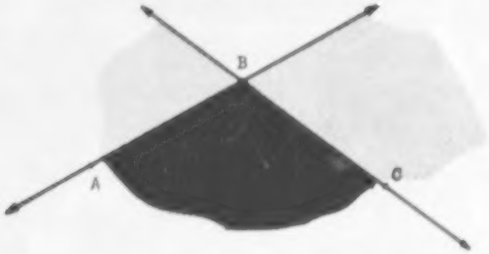
Exercises 1-8b

1. (a) $\angle XZY$, $\angle YZX$, $\angle XZV$, $\angle VZX$
$\angle WZY$, $\angle YZW$, $\angle WZV$, $\angle VZW$

(b) Z

(c) $\overrightarrow{ZX}$ or $\overrightarrow{ZW}$ and $\overrightarrow{ZY}$ or $\overrightarrow{ZV}$

2.



The doubly shaded part represents the interior of $\angle ABC$.

3. (a) $\angle PQR$ or $\angle RQP$

(b) $\{Q\}$

4. (a) false (b) false (c) false
(d) true (e) false

5. (a) $\angle SPT$, $\angle SPR$, $\angle SPQ$

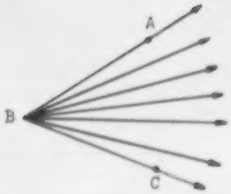
(b) $\overrightarrow{PQ} \cup \overrightarrow{PS} = \angle QPS$

(c) The intersection is the interior of $\angle RPS$.

6. (a) $\overrightarrow{OB}$ is also between $\overrightarrow{OX}$ and $\overrightarrow{OC}$.

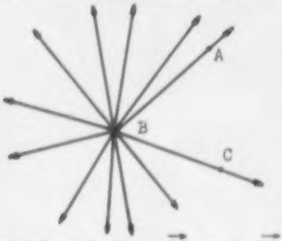
(b) $\overrightarrow{OX}$ is also between $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

7. (a)



(b) The figure formed would be the union of ∠ABC and its interior.

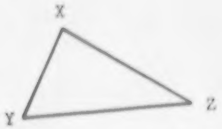
8. (a)



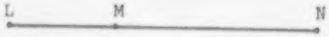
(b) The figure formed excluding $\overrightarrow{BA}$ and $\overrightarrow{BC}$, would be the exterior of ∠ABC.

Exercises 1-8c (Class Discussion)

1. (a)



(b)



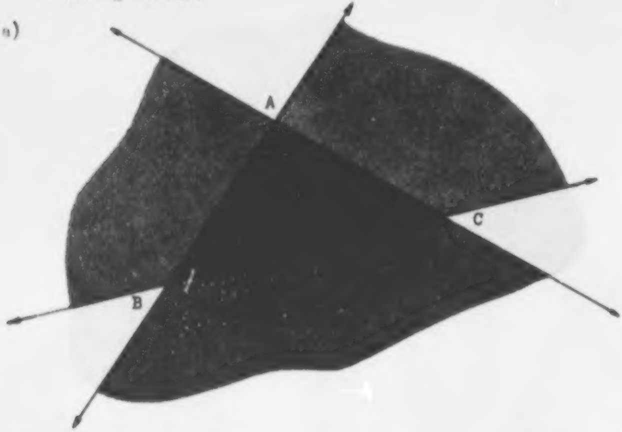
(c) Yes. △XYZ is a plane figure because the segments $\overline{XY}$, $\overline{YZ}$, and $\overline{XZ}$ lie in the plane determined by the three non-collinear points X, Y, Z.

(d) Triangle XYZ is the union of the segments $\overline{XY}$, $\overline{YZ}$, and $\overline{ZX}$ (X, Y, Z are non-collinear).

(e) Yes. A triangle is a simple closed path.

It can be traced starting at a point and ending at the same point without lifting the pencil from the paper and without the path crossing itself.

2. (a)



(b) The doubly shaded part represents the interior of $\angle ACB$.

(c) The part shaded three times is the interior of $\triangle ABC$.

(d) For three non-collinear points A, B, C, the interior of $\triangle ABC$ is the intersection of the following three half-planes; the A-side of $\overleftrightarrow{BC}$, the B-side of $\overleftrightarrow{AC}$, and the C-side of $\overleftrightarrow{AB}$.

(e) Yes. $\overline{PQ}$ will always intersect $\triangle ABC$.

1-9. Locating Positions and Points

In this section, a way of locating and plotting points in a plane using pairs of numbers is developed. At the outset, a seating chart is introduced to show how students may be located by a pair of numbers. If you are accustomed to considering rows horizontally instead of vertically on the chart, please feel free to interchange the designation of row and seat on the diagram. We are told that most seating charts use the designation of row and seat as shown in the text. The important idea is to use the pair of numbers in the same way as is done in locating points in a plane.

The step-by-step method of showing how a point S is located by a pair of numbers should be understood by all students. The number on the X-axis is always named first. The order of the numbers is important because if the order of a pair of two different numbers is changed then a different point is located. The term "ordered pair" is not introduced at this early stage.

Lots of practice should be provided for the students if necessary. It is noted that the plotting and locating of points is restricted to only a part of the plane in the beginning.

Exercises 1-9a (Class Discussion)

1. (a) Eve (b) Nell (c) (2,4) (d) (4,2)

2. {Ann (1,2), Kay (1,5), Mary (2,1), Emma (2,2), Jane (3,1), June (3,3), Nora (3,5), Myra (4,3), May (4,4), Eve (4,5), Sue (5,1), Nell (5,4)}

3. (a) These students sit in the 4th seat in each row.
   (b) Fred (1,4), Pete (2,4), Gary (3,4), May (4,4), Nell (5,4).
   (c) The second number in each pair is 4.

4. Jane (3,1), Bill (3,2), June (3,3), Gary (3,4), Nora (3,5).
   The first number in each pair is 3.
   These students sit in the 3rd row.

5. (a) The intersection of the two sets of students consists of Gary.
   (b) The intersection of the two sets of number pairs consists of the number (3,4).

6. (a) {John, Emma, June, May, Carl}
   (b) They sit diagonally from left-front (1,1) to right-rear (5,5) of the room.

7. (a) {Ann (1,2), Ray (1,3), Fred (1,4), Kay (1,5), Ed (2,3), Pete (2,4), Mike (2,5), Gary (3,4), Nora (3,5), Eve (4,5)}
   (b) They sit to the left of the diagonal from left-front to right-rear of the room.

Exercises 1-9b

1.

Point	A	B	C	D	E	F	G	H	I
x	3	2	10	9	4	2	3	9	9
y	2	3	2	4	4	2	3	1	3

2.

Point	A	B	C	D	E	F	G	H	I
Number Pair	(3,2)	(2,3)	(10,2)	(9,4)	(4,4)	(2,2)	(3,3)	(9,1)	(9,3)

3. (a) Points B, G, and I lie on the horizontal line through 3 on the Y-axis.

   (b) The second number for each number pair is 3.

4. (a) Points E, F, and G have number pairs in which the first number and second number are equal.

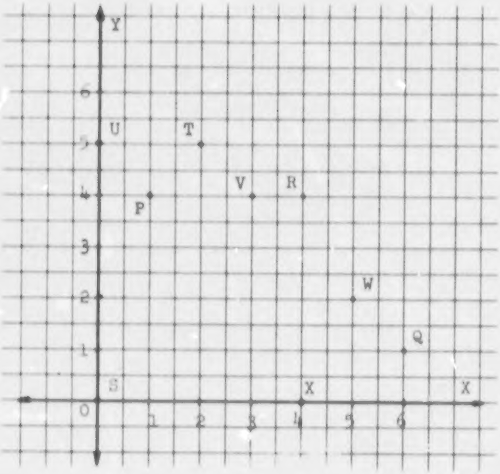
   (b) The point (91,91) also has a first number and second number which are equal.

   (c) The point (91,91) lies on the diagonal line which passes through points E, F, and G.

5. (a) Points D, H, and I lie on the vertical line through 9 on the X-axis, and they have number pairs in which 9 is the first number.

   (b) x = 9

6.



7. (a) Points Q, T, V, and W lie on a line. They each have number pairs in which the sum of the first number (x) and the second number (y) is 7. $x + y = 7$

(b) $1 + y = 7$, therefore, y must equal 6. $(1,y) = (1,6)$

8. (2,4)

9. For any point on the X-axis, the second number (y) in its number pair must be zero. $(y = 0)$ Thus, $(x,0)$ is the number pair for any point on the X-axis.

1-10. Coordinates

In order to continue plotting and locating points in other parts of the plane it is necessary to extend the assigning of numbers to points on the X- and Y-axes. The negative numbers are brought in as "opposites" of the already assigned positive numbers. This ties in nicely with the fact that positive and negative numbers are assigned to points on opposite half-lines. This also anticipates the development of opposites for integers in a later chapter.

It is implied that each point on the line is assigned a number, therefore we are actually "bringing in" the real numbers which include both the rational and irrational numbers. For the time being we shall refer to them simply as numbers that are called coordinates. Later, after the students have studied the various kinds of numbers, the process of coordinatizing a line and a plane can be explained more fully.

Exercises 1-10

1. The process is the same as for the X-axis. Draw a vertical line and assign 0 to the point of intersection with the X-axis.

   Select a unit point upward and assign it the number 1. Mark off points at unit intervals upward and assign the whole numbers in succession. It is assumed that numbers can be assigned to all the unmarked points above 0.

   Now mark off the same unit intervals downward and label with the negative numbers $^-1$, $^-2$, $^-3$, and so on in succession. Again it is assumed that negative numbers can be assigned to all the unmarked points below 0 as "opposites" of the already assigned positive numbers.

81

2.

Point	A	B	C	D	E
Coordinate	$^-2$	$\frac{1}{2}$	$1\frac{2}{3}$ or $\frac{5}{3}$	3	6

3.

Point	T	K	Q	L	O
Coordinate	$\frac{3}{2}$	$^-3$	2.8	$^-1$	0

1-11. Coordinates in the Plane

Now that both positive and negative numbers are assigned to points on the X- and Y-axes, the process of plotting and locating points in a plane is continued. The process is the same as the one described in the previous section for points located by pairs of whole numbers.

For each pair of numbers (x,y) the first number is called the X-coordinate (x) and the second number is called the Y-coordinate (y). The coordinate plane is defined as a plane in which each point is assigned a pair of coordinates, and each pair of coordinates designates a point. In other words, there is a one-to-one correspondence between points in the plane and pairs of coordinates.

The coordinate axes separate the plane into four parts called quadrants. These quadrants can be defined as intersections of half-planes. It would be time well spent with the students to analyze each quadrant in the same manner as is done in the students' text for Quadrant II. Students should develop skill in telling in what quadrant a point is located by glancing at its coordinates.

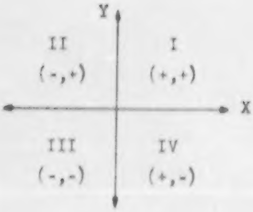
Exercises 1-11

1.

Point	A	B	C	D	E	F	G
Coordinates	$(3,2)$	$(2,3)$	$(^-3,2)$	$(2,^-3)$	$(^-2,3)$	$(3,^-2)$	$(^-3,^-2)$

Point	H	I	J	K	L	M	N
Coordinates	$(^-2,^-3)$	$(7,^-5)$	$(9,4)$	$(^-2,^-5)$	$(^-2,2)$	$(10,2)$	$(0,2)$

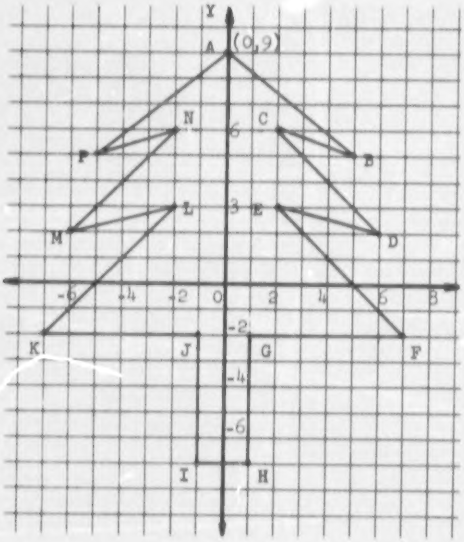
2.


Y
II
(-,+)
I
(+,+)
X
III
(-,-)
IV
(+,-)

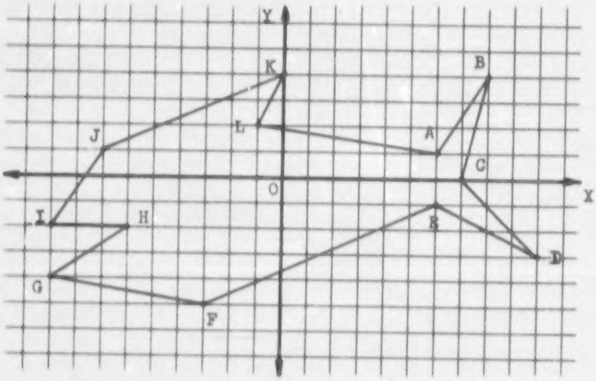
Quadrant III (-,-)

Quadrant IV (+,-)

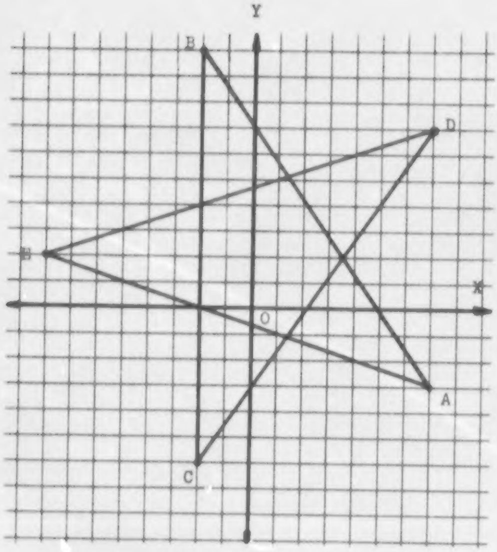
3.


Y
A
(0,9)
N
C
6
P
B
L
3
E
M
D
-6
-4
-2
0
2
4
6
8
-2
K
J
G
F
-4
-6
I
H

4.


Y
K
B
J
L
A
C
0
X
I
H
E
D
G
F

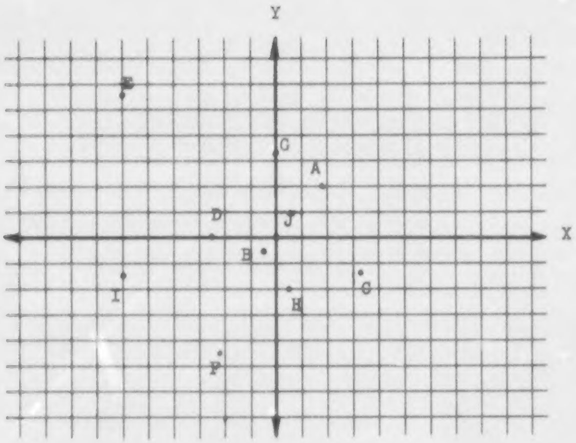
5.



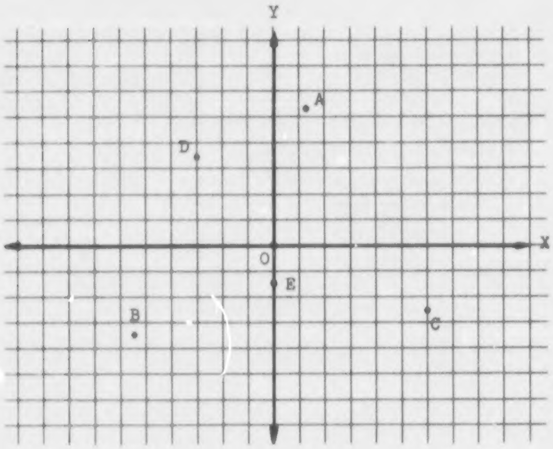
6.

	Point	Quadrant		Point	Quadrant
(a)	(3,5)	I	(g)	$(^-3,^-5)$	III
(b)	$(1,^-4)$	IV	(h)	(0,2)	none
(c)	$(^-4,^-4)$	III	(i)	$(31,^-31)$	IV
(d)	$(^-3,^-1)$	III	(j)	$(^-7,^-72)$	III
(e)	(8,6)	I	(k)	$(^-51,25)$	II
(f)	$(^-7,1)$	II	(ℓ)	$(^-27,^-100)$	III

7.


Y
E
G
A
D
J
X
B
I
C
H
F

8.



1-12. Graphs in the Plane

The purpose of this last section is to provide some opportunity for students to examine some patterns of coordinates of points that lie on the same line. The development is very informal and exploratory.

The lines are described very informally as being horizontal or vertical through certain points on the Y- or X-axis. The only equation of a line introduced is $y = x$ in the class discussion exercises.

There will be ample time later for students to develop equations for lines, so at this stage they are asked only to look for patterns.

Exercises 1-12a (Class Discussion)

1.

Point	A	B	C	D	E	F
X-coordinate	0	5	2	$^{-}4$	$^{-}2$	4
Y-coordinate	0	5	2	$^{-}4$	$^{-}2$	4

2. For each point, the X- and Y-coordinates are the same.

3.

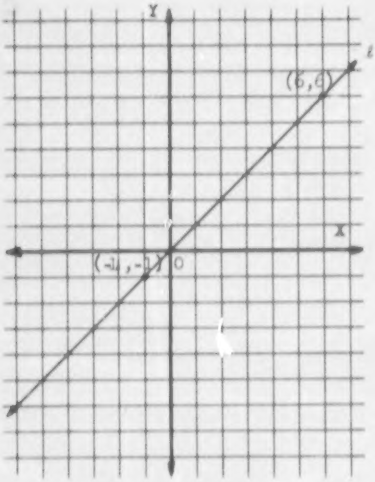
Point	G	H	J	K	L	M	N
X-coordinate	$3\frac{1}{2}$	$^{-}3\frac{1}{2}$	$\frac{1}{3}$	$1\frac{2}{3}$	$^{-}2\frac{1}{2}$	$-\frac{1}{2}$	$^{-}4\frac{1}{2}$
Y-coordinate	$3\frac{1}{2}$	$^{-}3\frac{1}{2}$	$\frac{1}{3}$	$1\frac{2}{3}$	$^{-}2\frac{1}{2}$	$-\frac{1}{2}$	$^{-}4\frac{1}{2}$

4. Yes

5. For each point on this line, the X- and Y-coordinates are equal. $y = x$

Exercises 1-12b

1.



(a) yes

(b) yes

(c) no

(d) no

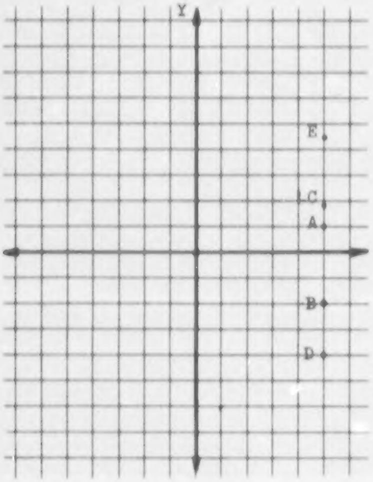
(e) a = b

(f) c ≠ d

(g) yes

(h) no

2.



(a) $A(5,1)$

(b) $B(5,{}^-2)$

(c) $C(5,\frac{5}{3})$

(d) $D(5,{}^-4)$

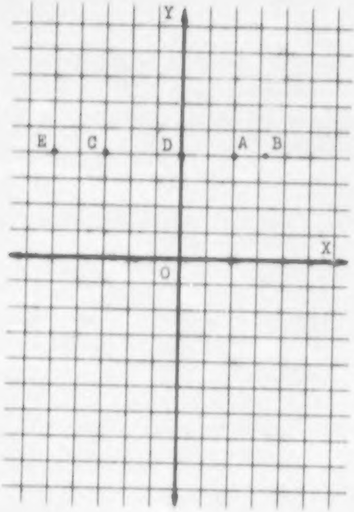
(e) $E(5,4\frac{1}{2})$

(f) The points lie on the vertical line through 5 on the X-axis.

(g) The graph of all points whose coordinates are $(5,y)$ is the vertical line through 5 on the X-axis.

90

3.



(a) A(2,4)

(b) B($3\frac{1}{3}$, 4)

(c) C(¯3,4)

(d) D(0,4)

(e) E(¯5,4)

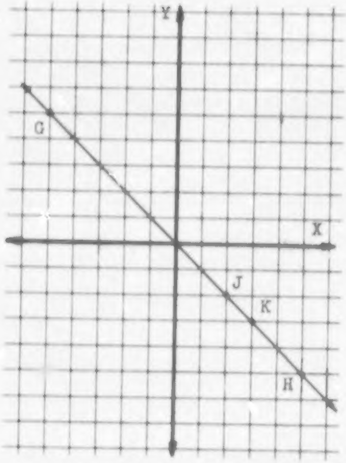
(f) The points lie on the horizontal line through 4 on the Y-axis.

(g) The graph of all points whose coordinates are (x,4) is the horizontal line through 4 on the Y-axis.

4. The point of intersection would be

(a) (5,5) (b) (4,4) (c) (5,4)

5.



(a) Y-coordinate of J is ¯2.

(b) X-coordinate of K is 3.

(c)

Point	L	M	N	P
X-coordinate	¯5	5	117	¯117
Y-coordinate	5	¯5	¯117	117

6. (a) no (b) yes (c) yes
(d) yes (e) yes (f) yes
(g) yes

Suggested Test Items

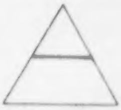
Multiple Choice

1. Let set A = {1,3,5,7,9}
and set B = {2,3,7,9}.
The union of sets A and B is:

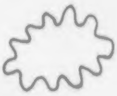
(A) {3,7,9}
(B) {1,3,5,7,9}
(C) {2,3,7,9}
(D) {1,2,3,5,7,9}
(E) {2}

2. Which one of the following is a simple closed path?

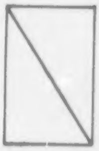
(A)



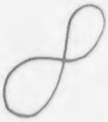
(C)



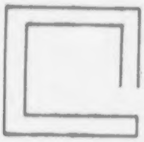
(E)



(B)



(D)



3. (See Figure 1) Point P separates line ℓ into two:

(A) rays
(B) segments
(C) half-lines
(D) half-planes
(E) half-spaces

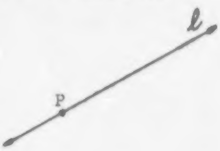


Figure 1

Questions 4-5 are based on Figure 2.

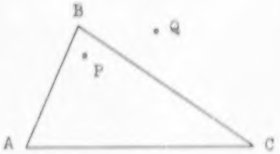


Figure 2

4. Points P and Q are both:

   (A) in the interior of $\triangle ABC$.
   (B) in the exterior of angle BCA.
   (C) on the same side of $\overleftrightarrow{BC}$.
   (D) on $\overline{AQ}$.
   (E) in the interior of angle BAC.

5. $\overleftrightarrow{PQ}$ intersects $\triangle ABC$ in:

   (A) no points
   (B) 1 point
   (C) 2 points
   (D) 3 points
   (E) a segment
   (F) an unlimited number of points.

Questions 6-9 are based on Figure 3. For each question choose your answer from the following list.

(A) $\overline{XV}$
(B) Z
(C) Y
(D) $\overline{XZ}$
(E) the empty set

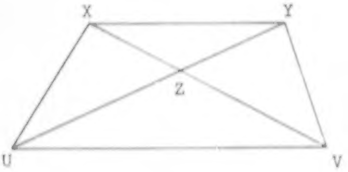


Figure 3

6. What is the intersection of $\overline{XV}$ and $\overline{UY}$? __________

7. What is $\overline{XV} \cup \overline{XZ}$? __________

8. What is $\overline{XU}$ $\overline{YV}$? __________

9. What is $\overrightarrow{UZ} \cap \overleftrightarrow{VY}$? __________

10. Angle YXZ (Figure 4) is a set of points consisting of the union of two:

(A) segments
(B) lines
(C) points
(D) half-lines
(E) rays

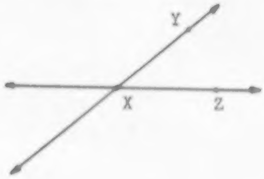


Figure 4

11. A plane is separated into two half-planes by:

(A) a point
(B) a line
(C) a simple closed path
(D) a plane
(E) an angle

Questions 12-14 are based on Figure 5.

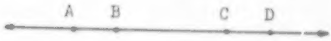


Figure 5

12. $\overrightarrow{AB} \cap \overrightarrow{BA}$ is:

(A) $\overline{AB}$
(B) $\overleftrightarrow{AB}$
(C) A, B
(D) $\overrightarrow{AB}$
(E) the empty set

13. $\overleftrightarrow{AC} \cap \overrightarrow{BC}$ is:

(A) $\overline{BC}$
(B) $\overrightarrow{BC}$
(C) $\overline{AC}$
(D) $\overleftrightarrow{AC}$
(E) $\overrightarrow{CB}$

14. $\overline{AC} \cup \overrightarrow{BC}$ is:

(A) $\overline{AC}$
(B) $\overleftrightarrow{AC}$
(C) $\overrightarrow{AC}$
(D) $\overline{BC}$
(E) $\overrightarrow{BC}$

15. A ray without its endpoint is called:

(A) a line
(B) a segment
(C) an angle
(D) a half-space
(E) a half-line

Questions 16-18 are based on Figure 6. For each question choose your answer from the following list.

(A) $\overline{UV}$ and $\overline{XY}$
(B) $\overline{UV}$ and $\overline{VX}$
(C) $\overline{UX}$ and $\overline{XY}$
(D) $\overline{UX}$ and $\overline{VY}$
(E) $\overline{UY}$ and $\overline{YZ}$

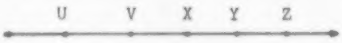


Figure 6

16. Name 2 segments whose intersection is a segment. __________

17. Name 2 segments whose intersection is empty. __________

18. Name 2 segments whose union is $\overline{UY}$. __________

Questions 19-22 are based on Figure 7.

19. The intersection of $\triangle ABC$ and $\overleftrightarrow{BC}$ is:

(A) $\overrightarrow{BC}$
(B) $\overline{BC}$
(C) $\overrightarrow{CB}$
(D) points B and C
(E) the empty set

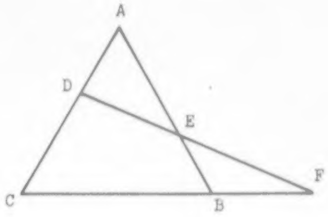


Figure 7

20. $\triangle ABC$ angle ACB is the same as:

(A) $\overline{AC} \cup \overline{CB}$

(B) $\overline{AC} \cap \overline{CB}$

(C) $\overleftrightarrow{AC} \cup \overleftrightarrow{CB}$

(D) C

(E) $\overrightarrow{CA} \cup \overrightarrow{CB}$

21. Triangles EBF and ADE intersect in:

(A) 1 point

(B) 2 points

(C) a line

(D) a segment

(E) a region

22. $\triangle ABC$ is the union of:

(A) $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$

(B) $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{BC}$

(C) $\overline{AB}$, $\overline{AC}$, $\overline{BC}$

(D) angle BAC, $\overline{BC}$

(E) A, B, C

23. In Figure 8 how many different segments are determined by the points P, Q, R, and S?

(A) 3

(B) 4

(C) 5

(D) 6

(E) 7

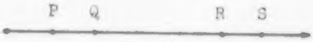


Figure 8

24. Let plane ABC be the plane of the chalkboard in your classroom, and let R and S name any two points of space on opposite sides of plane ABC. Then, the intersection of plane ABC and segment RS is:

(A) a line

(B) a point

(C) two points

(D) a segment

(E) the empty set

25. The point with coordinates (0,0) is located in the:

(A) upper half-plane

(B) lower half-plane

(C) left half-plane

(D) right half-plane

(E) none of these

(F) all of these

26. The point with coordinates $(47, {}^{-}47)$ lies in the:

(A) upper half-plane
(B) lower half-plane
(C) right half-plane
(D) left half-plane
(E) both right and lower half-planes
(F) none of these

Completion

1. Each of the following objects suggests a point, a line, or a plane. Supply the correct word.

(a) the floor of this room. ______________
(b) a button on a shirt. ______________
(c) the wall of a house. ______________
(d) a fishing rod. ______________
(e) a thumb tack on a bulletin board. ______________
(f) a sail on a sailboat. ______________

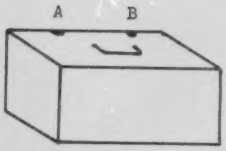
2. Each of the following objects suggests a point, a line, or a plane. Fill in the correct word.

(a) a typewriter key. ______________
(b) the school gymnasium floor. ______________
(c) a traffic light. ______________
(d) a space bar on a typewriter. ______________
(e) a screen on a window. ______________
(f) a parking lot. ______________
(g) a baton. ______________

3. Let A name a point on the floor of this room, and let B name a point on the ceiling of this room.

(a) How many lines may be drawn containing points A and B? ________
(b) State the property which justifies your answer. ______________
____
____

4. The figure on the right represents a tool box with hinges for the top at points A and B.



(a) How many different planes are suggested by the door as it swings up? ____________

(b) How many different planes contain points A and B? ____________

(c) State the property that justifies your answer to (b). ____________

____

____

5. Consider the following points suggested by

X, a foot of your chair
Y, the top of your head
Z, the knob on the door

(a) How many planes are suggested by these points? ____________

(b) State the property that justifies your answer. ____________

____

____

6. A, B, C, and D are four distinct points, not all in the same plane. Also, no three of these points are collinear.

(a) How many different planes are determined by these four points?

____

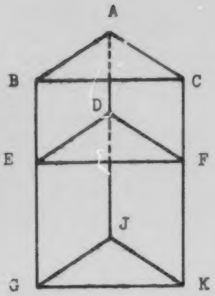
(b) Name the planes. ____________

(c) Do any pair of planes named in part (b) intersect? ____________

(d) If so, how many pairs of planes intersect? ____________

(e) Draw the figure formed by the intersections of the planes.

7. The diagram at the right represents a corner cupboard with a shelf.



(a) Name two skew lines. ________

(b) Name two parallel planes. ________

(c) Name two intersecting lines.

____

(d) Name their intersection. ________

(e) Name two parallel lines. ________

(f) Name two intersecting planes.

____

(g) Name their intersection. ________

(h) What is $\overleftrightarrow{AB} \cap \overleftrightarrow{CF}$? ________

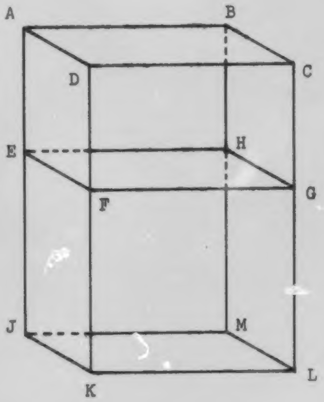
(i) What is plane EDF ∩ plane ACK? ________

(j) What is plane ABC ∩ plane GJK? ________

(k) What is $\overleftrightarrow{AJ} \cap \overleftrightarrow{FD}$? ________

(ℓ) What is plane AJG ∩ $\overleftrightarrow{ED}$? ________

8. The diagram at the right represents a bookcase with a shelf.



(a) Name two parallel lines.

____

(b) Name two skew lines. ________

(c) Name two parallel planes.

____

(d) Name two intersecting planes.

____

(e) Name the intersection of the two planes mentioned in part (d).

____

(f) Name two intersecting lines.

____

(g) Name the intersection of the two lines mentioned in part (f).

____

(h) What is $\overleftrightarrow{EF} \cap \overleftrightarrow{BH}$? ________

(i) What is $\overleftrightarrow{HG}$ ∩ plane BLM?

9. See the figure at the right.

(a) $\overline{AF} \cap \overline{BF}$. ________

(b) $\overline{HK} \cap \overline{FC}$. ________

(c) $\overline{KF} \cap \overline{GB}$. ________

(d) $\overline{KF} \cup \overline{FH}$. ________

(e) $\overline{BE} \cup \overline{AG}$. ________

(f) Name three collinear points.

____

(g) $\overleftrightarrow{EG} \cap \overline{BG}$. ________

(h) $\overline{DF} \cup \overline{KH}$. ________

(i) $\overline{CD} \cap \overline{AB}$. ________

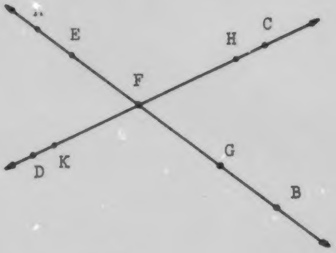
(j) $\overleftrightarrow{CD} \cap \overleftrightarrow{AB}$. ________

(k) Name two opposite rays.

____

(l) $\overrightarrow{FE} \cap \overrightarrow{FK}$. ________

(m) $\overrightarrow{FK} \cup \overleftrightarrow{DC}$. ________



10. In the figure at the right, find:

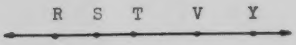
(a) $\overline{RS} \cap \overline{SV}$. ________

(b) $\overline{RT} \cap \overline{SV}$. ________

(c) $\overline{RS} \cap \overline{TV}$. ________

(d) $\overline{RS} \cup \overline{SY}$. ________

(e) $\overline{RT} \cup \overline{SV}$. ________



11. See the figure at the right.

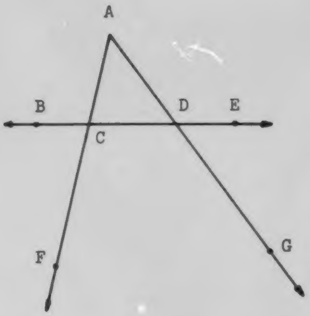
(a) Name three collinear points.

____

(b) Find $\overleftrightarrow{AF} \cap \overline{AD}$. ________

(c) Find $\overline{DG} \cap \overline{CF}$. ________

(d) Find $\overline{BD} \cup \overline{CE}$. ________



12. See the figure at the right.

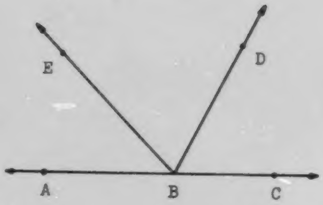
(a) Find $\overrightarrow{AB} \cup \overrightarrow{AC}$. ________

(b) Name two opposite rays.

____

(c) Find $\overrightarrow{EB} \cap \overrightarrow{DB}$. ________

(d) Find $\overrightarrow{BC} \cup \overrightarrow{AC}$. ________



13. Complete each of the following:

(a) A point separates a ________ into two ________.

(b) A line separates a ________ into two ________.

(c) A plane separates ________ into two ________.

14. What are the boundaries of each of the following?

(a) a half-line. ________

(b) a half-plane. ________

(c) a half-space. ________

15. Explain how each of the following may be regarded as a separation figure:

(a) the midfield stripe on a football field. ________

(b) a net on a tennis court. ________

(c) a pencil point. ________

16. Which of the following figures have boundaries? If the figure has a boundary identify the boundary.

(a) a line ________

(b) a half-plane ________

(c) space ________

(d) a half-line ________

(e) a half-space ________

(f) a plane ________

17. (a) Into how many non-overlapping regions do three parallel lines separate a plane? ________

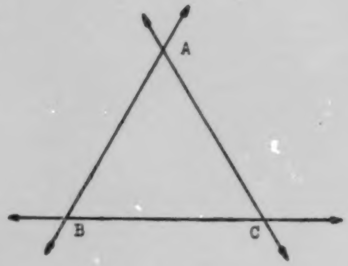
(b) Into how many non-overlapping regions do two intersecting lines separate a plane? ________

18. (a) Into how many non-overlapping regions does the union of the segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$ separate the plane?

____

(b) Into how many non-overlapping regions does the union of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$ separate the plane?

____



19. A triangle separates its plane into three sets of points. Describe these sets of points.

(a) ____

(b) ____

(c) ____

20. An angle separates its plane into three sets of points. They are:

(a) ______________________

(b) ______________________

(c) ______________________

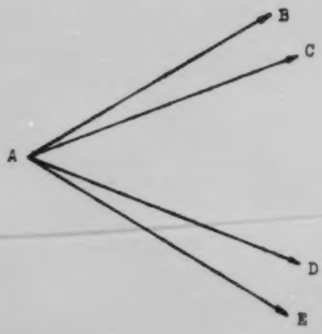
21. See the figure at the right.

(a) What is $\angle BAC \cup \angle CAD$?

____

(b) What is $\angle BAC \cap \angle CAD$?

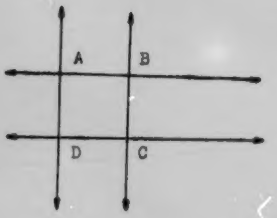
____



22. (a) Into how many non-overlapping regions does the union of the segment $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ separate the plane?

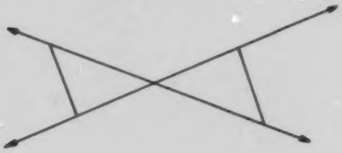
____

(b) Into how many non-overlapping regions does the union of the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CD}$, and $\overleftrightarrow{DA}$ separate the plane? ____________________

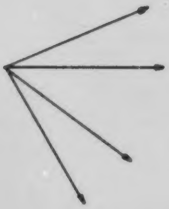


23. Into how many non-overlapping regions is the plane separated by each of the following figures?

(a)



(b)



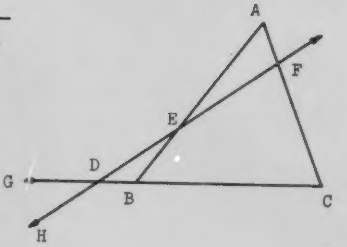
24. See the figure at the right.

(a) What is $\overleftrightarrow{DE} \cap \triangle ABC$? ________

(b) Name a point on the A-side of $\overleftrightarrow{BC}$. ________

(c) Name a point in the exterior of $\triangle ABC$. ________

(d) What is $\overrightarrow{BC} \cup \overrightarrow{BE}$? ________



25. See the figure at the right.

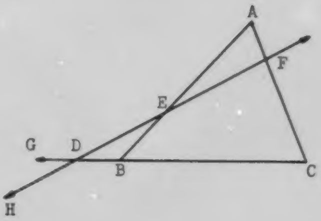
(a) What is $\overleftrightarrow{EG} \cap \triangle ABC$?

____

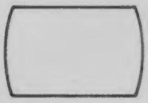
(b) Name a point on the G-side of $\overleftrightarrow{BC}$. ________________

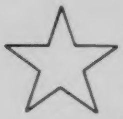
(c) Name a point in the exterior of $\triangle ABC$. ________________

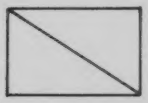
(d) What is $\overrightarrow{GE} \cup \overrightarrow{GC}$? ____________

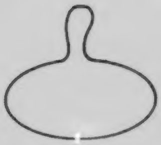


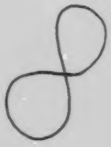
26. Which of the following figures are simple closed paths?


(a)

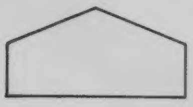

(b)

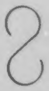

(c)

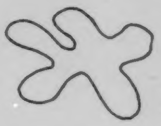

(d)

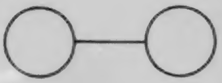

(e)

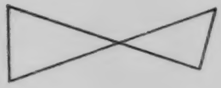
27. Which of the following are simple closed paths?


(a)


(b)


(c)


(d)


(e)

28. In a coordinate plane, the point of intersection of the horizontal and vertical number lines is called the ______________.

29. In a coordinate plane, the horizontal number line is named the _____ axis, and the vertical number line is named the _____ axis.

30. In a coordinate plane, the intersection of the half-plane below the horizontal axis and the half-plane to the right of the vertical axis is defined to be quadrant ________.

31. The word quadrant refers to one of __________ (How many) distinct regions in a coordinate plane.

32. Each point in a coordinate plane has __________ (How many) numbers associated with it.

33. What are the coordinates of the origin? ______________

34. If a point belongs to the x-axis, then its y-coordinate must be ______________.

35. Write a mathematical sentence describing the line such that the x-coordinate of each point on the line is equal to its y-coordinate?

____

36. Identify the location of the points whose coordinates are shown below. Choose your answer from the following list.

quadrant I
quadrant II
quadrant III
quadrant IV
horizontal axis
vertical axis

(a) $(4,7)$ ______________
(b) $({}^{-}5,{}^{-}4)$ ______________
(c) $({}^{-}9,0)$ ______________
(d) $(\frac{1}{4},\frac{1}{4})$ ______________
(e) $(6,{}^{-}4.5)$ ______________
(f) $(0,{}^{-}45)$ ______________
(g) $({}^{-}1.4,4.85)$ ______________
(h) $(3,{}^{-}5\frac{1}{2})$ ______________

37. Given the following ordered pairs of numbers, write the number of the quadrant in which you find the point corresponding to each of these ordered pairs.

	Ordered Pair	Quadrant
(a)	$(3,5)$	_______
(b)	$(1,{}^{-}4)$	_______
(c)	$({}^{-}4,4)$	_______
(d)	$({}^{-}3,{}^{-}1)$	_______
(e)	$(8,6)$	_______
(f)	$(7,{}^{-}1)$	_______
(g)	$({}^{-}3,{}^{-}5)$	_______

38. (a) Both numbers of the ordered pair of coordinates are positive. The point is in quadrant _____.

(b) Both numbers of the ordered pair of coordinates are negative. The point is in quadrant _____.

(c) The x-coordinate of an ordered pair is negative and the y-coordinate is positive. The point is in quadrant _____.

(d) The x-coordinate of an ordered pair is positive and the y-coordinate is negative. The point is in quadrant _____.

39. (a) If the x-coordinate of an ordered pair is zero and the y-coordinate is not zero, where does the point lie? ____________________

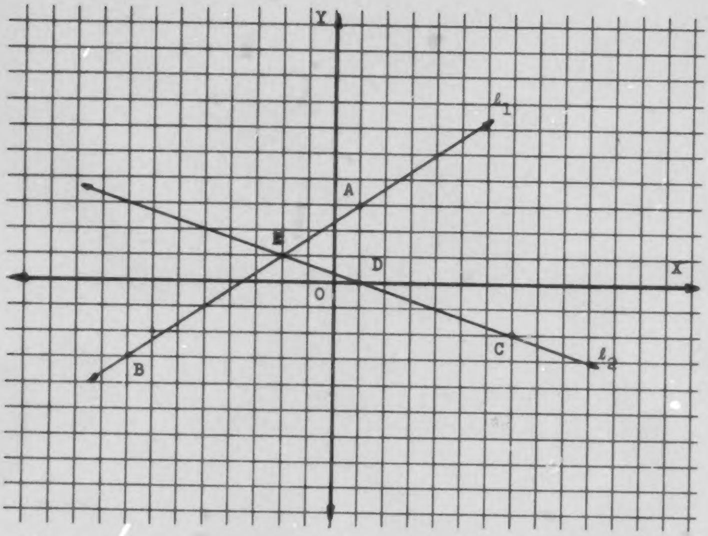
(b) If the x-coordinate of an ordered pair is not zero and the y-coordinate is zero, where does the point lie? ______________

(c) If both coordinates of an ordered pair are zero, where is the point located? ____

(d) Do the points on either the X-axis or the Y-axis lie in any of the four quadrants? _________ Why, or why not? ______________

____

40.



(a) The coordinates of point A are ______________.

(b) The coordinates of point B are ______________.

(c) The coordinates of point C are ______________.

(d) The coordinates of point D are ______________.

(e) The coordinates of the point of intersection of line ℓ_1 and line ℓ_2 are ______________.

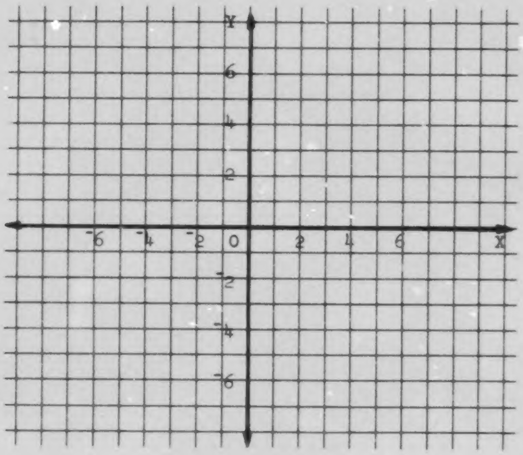
(f) Name the point which is located in Quadrant III. ______________

(g) Name the point which is located in Quadrant IV. ______________

Problems

1. Plot the points corresponding to the following pairs of numbers.

   (a) (2,5)
   (b) (6,2)
   (c) (0,3)
   (d) (3,0)
   (e) $(^-5,1)$
   (f) Draw a line through the points whose coordinates are (2,5) and $(^-5,1)$.
   (g) Draw a line through the points whose coordinates are (0,3) and (3,0).
   (h) In which quadrant do these lines intersect?



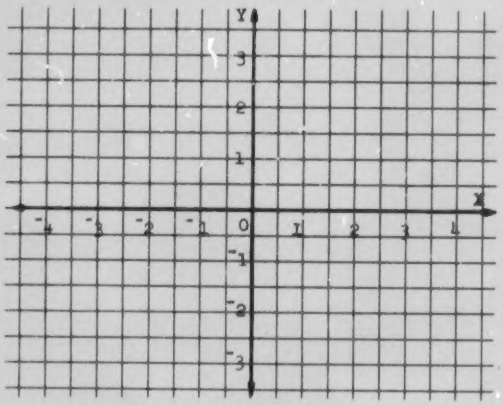
2. (a) Plot the points of set $S = \{A(2,1), B(^-2,1), C(^-2,^-3), D(2,^-3)\}$.

(b) Use a straight edge to join A to B, B to C, C to D, and D to A.

(c) What kind of figure is formed?

(d) Draw the diagonals of the figure.

(e) The coordinates of the point of intersection of the diagonals seem to be ______________________________.



Answers to Suggested Test Items

Multiple Choice

1.	D	12.	A
2.	C	13.	B
3.	C	14.	C
4.	E	15.	E
5.	C	16.	D
6.	B	17.	A
7.	A	18.	C or D
8.	E	19.	B
9.	C	20.	A
10.	E	21.	E
11.	B	22.	C

23. D

24. B

25. E

26. E (B and C are also correct.)

Completion

1. (a) plane
   (b) point
   (c) plane
   (d) line
   (e) point
   (f) plane

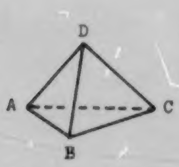
2. (a) point
   (b) plane
   (c) point
   (d) line
   (e) plane
   (f) plane
   (g) line

3. (a) one
   (b) Property 1: Through any two different points in space there is exactly one line.

4. (a) an unlimited number
   (b) an unlimited number
   (c) Property 3: If there are two points, then many planes contain this pair of points.

5. (a) one
   (b) Property 4: Through any three points, not all on the same line, there is exactly one plane.

6. (a) 4
   (b) plane ABC, plane ABD, plane ACD, plane BCD
   (c) Yes
   (d) 6
   (e)



7. (a) $\overleftrightarrow{AB}$ and $\overleftrightarrow{CK}$
   (b) plane ABC and plane DEF
   (c) $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$
   (d) A
   (e) $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$
   (f) plane ABG and plane ACK
   (g) $\overleftrightarrow{AJ}$
   (h) $\emptyset$ (the empty set)
   (i) $\overleftrightarrow{DF}$
   (j) $\emptyset$ (the empty set)
   (k) D
   (ℓ) $\overleftrightarrow{ED}$

8. (a) $\overleftrightarrow{DK}$ and $\overleftrightarrow{CL}$
   (b) $\overleftrightarrow{DK}$ and $\overleftrightarrow{BC}$
   (c) plane ABC and plane JML
   (d) plane ABC and plane DCL
   (e) $\overleftrightarrow{DC}$
   (f) $\overleftrightarrow{DK}$ and $\overleftrightarrow{KL}$
   (g) K
   (h) $\emptyset$ (the empty set)
   (i) $\overleftrightarrow{HG}$

9. (a) F
(b) $\overline{FH}$
(c) $\emptyset$ (the empty set)
(d) $\overline{KH}$
(e) $\overline{AB}$
(f) A, E, F
(g) $\overline{BG}$
(h) $\overline{DH}$
(i) F
(j) F
(k) $\overrightarrow{FD}$ and $\overrightarrow{FC}$
(ℓ) F
(m) $\overleftrightarrow{DC}$

10. (a) S
(b) $\overline{ST}$
(c) $\emptyset$ (the empty set)
(d) $\overline{RY}$
(e) $\overline{RV}$

11. (a) A, C, F
(b) A
(c) $\emptyset$ (the empty set)
(d) $\overline{BE}$

12. (a) $\overrightarrow{AB}$ on $\overrightarrow{AC}$
(b) $\overrightarrow{BA}$ and $\overrightarrow{BC}$
(c) B
(d) $\overrightarrow{AC}$

13. (a) line, half-lines
(b) plane, half-planes
(c) space, half-spaces

14. (a) a point
(b) a line
(c) a plane

15. (a) models a line
(b) models a plane
(c) models a point

16. (a) no boundary
(b) a line
(c) no boundary
(d) a point
(e) a plane
(f) no boundary

17. (a) 4
(b) 4

18. (a) 2
(b) 7

19. (a) the triangle
(b) the interior of the triangle
(c) the exterior of the triangle

20. (a) the angle
(b) the interior of the angle
(c) the exterior of the angle

21. (a) $\angle BAC \cup \angle CAD$
(b) $\overrightarrow{AC}$

22. (a) 2
(b) 9

23. (a) 6
(b) 4

24. (a) $\{E,F\}$
(b) E
(c) D
(d) $\angle CBE$

25. (a) $\{F,G\}$
(b) A
(c) D
(d) $\angle EGC$

26. a, b, d

27. a, c

28. origin

29. X, Y

30. IV

31. 4

32. 2

33. (0,0)

34. zero

35. y = x

36. (a) Quadrant I
    (b) Quadrant III
    (c) horizontal axis
    (d) Quadrant I
    (e) Quadrant IV
    (f) vertical axis
    (g) Quadrant II
    (h) Quadrant IV

37. (a) I
    (b) IV
    (c) II
    (d) III
    (e) I
    (f) IV
    (g) III

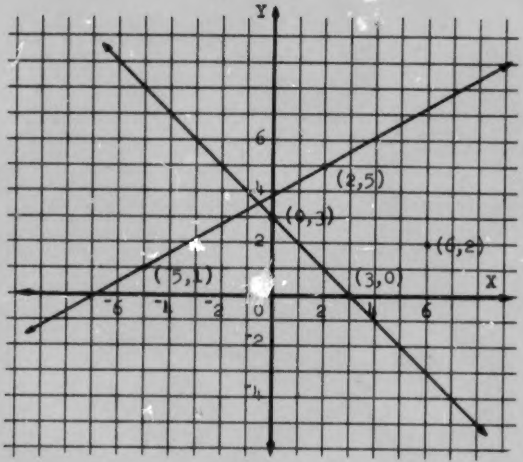
38. (a) I
    (b) III
    (c) II
    (d) IV

39. (a) on the Y-axis excluding the origin
    (b) on the X-axis excluding the origin
    (c) at the origin
    (d) No. The word quadrant refers to the interior of any of the 4 right angles formed by the two axes.

40. (a) (1,3)
    (b) $(^-8,^-3)$
    (c) $(7,^-2)$
    (d) (1,0)
    (e) $(^-2,1)$
    (f) B
    (g) C

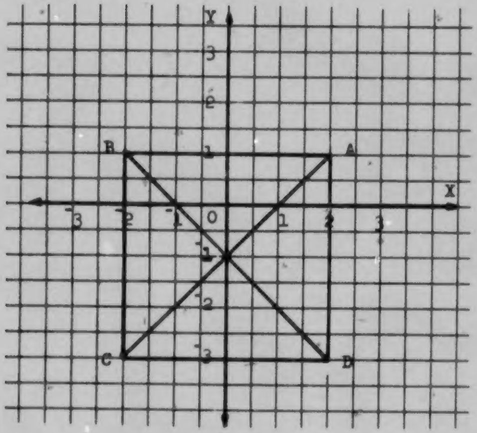
Problems

1. (a), ..., (g)



(h) Quadrant II

2. (a), (b), (d).



(c) a square

(e) $(0,^{-}1)$

Chapter 2

FUNCTIONS

2-1. Car Travel

The relationship between the distance traveled, the speed, and the time, is familiar to everyone who has traveled by car. For example, if you could maintain a speed of exactly 50 miles per hour, you could travel 50 miles in one hour, 100 miles in two hours, and so on. A shorter way of saying this would be:

distance traveled = (miles per hour) × (number of hours).

In terms of the situation mentioned above you could simply write:

$$d = 50 \cdot t,$$

where d represents the number of miles traveled, 50 represents the speed in miles per hour, and t represents the number of hours.

Another way of describing this relationship is to say,

> "The distance traveled depends on the number of hours you have been traveling."

In present day language we would say that,

> "The distance traveled is a function of the time spent in traveling."

The rule, or formula, $d = 50 \cdot t$, says that, if you input a value for t and do the indicated calculation, then you output exactly one corresponding numerical value for d. It is this kind of relationship, called a function, that we want to discuss in this chapter.

Exercises 2-1a

(Class Discussion)

1. Complete the table of values below. For each t we have a distance, 50t, which corresponds to the input t.

Input	Output
t	50t
0	0
1	50
2	?
3	?
4	?
5	250

New York City
250 miles

Figure 1

When we use the relationship between d and t, we are actually using a model of the real situation for which a number of necessary details such as heavy traffic, gas stops, and meals seem to be ignored. In the actual planning of an automobile trip, however, we often do estimate our speed and how far we can travel by such a model. You have probably heard people make such remarks as, "You can average 65 miles an hour on that stretch of highway," or "You'll be lucky to average 25 miles an hour on that narrow, winding road." Do these statements mean that the first speaker kept an exact speed of 65 miles per hour and the second speaker [illegible] able to drive always at 25 miles an hour? Perhaps, but this is not often true.

It is possible that the trip of 250 miles might have been as follows:

Hour of travel	No. of miles traveled	Comment
1st	65	Interstate Highway had very little traffic.
2nd	50	A different highway had a few traffic lights.
3rd	20	A quick lunch and gas stop was necessary.
4th	60	A return to the Interstate highway followed.
5th	55	The traffic began to get heavy.

That is, if the trip began at noon, by one o'clock the car would have traveled 65 miles; by two o'clock, 50 more miles for a total of 115 miles. The average speed after two hours would be $57\frac{1}{2}$ miles per hour.

Exercises 2-1b

(Class Discussion)

1. Putting the above information in another form, add to this information by completing the following table:

	(m/p/h)	(Total distance)	(Average speed)
Time	No. miles added each hour	No. miles of total travel at end of each hour	No. miles per hour averaged at end of each hour
t = 1	65	d = 65	65
t = 2	50	d = 115	$57\frac{1}{2}$
t = 3	20	d = 135	?
t = 4	60	?	?
t = 5	55	?	50

2. Another example of a functional relationship is the correspondence between the number of the year and a person's age in years. This relationship can be shown on a graph. We show below the graphs of the ages of two brothers during a ten-year period. Bob was exactly 8 years old when John was born in 1950. Just recently, the brothers were wondering about the relationship of their ages at different times. Can you answer their questions from the graphs given below?

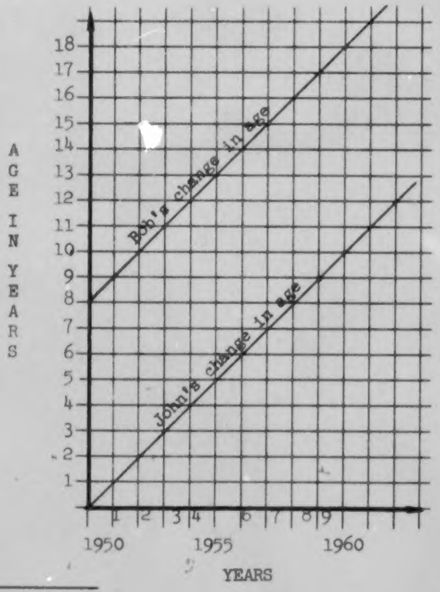
(a) Can Bob and John ever be the same age?

(b) Was Bob ever twice the age of John? If so, when?

(c) Was Bob ever three times the age of John? If so, when?

(d) Was Bob ever five times the age of John? If so, when?

(e) Was Bob ever nine times the age of John? If so, when?



In Exercise 1 above, the correspondence between the "time" and the "average speed" was easily shown in a table. This method of representing a function is very often the only way that such information can be shown.

In Exercise 2 an important reason for studying functional relationships is illustrated. Namely, that by "looking at" functions you can find additional, important information that usually is not apparent in the original form of the given situation. In this situation the representation of the functions by graphs should have made it easier to discover the answers to the questions.

2-2. Falling Objects

Near the end of the sixteenth century (around 1590), as the story goes, Italian mathematician and scientist, Galileo, performed an important experiment in the city of Pisa, which is located on the Arno River in central Italy. He hauled a ten-pound weight and a one-pound weight to the top of the now famous leaning tower of Pisa and dropped them off at the same time. Friends of Galileo on the ground below observed that the weights remained side by side as they fell and that they seemed to hit the ground at exactly the same instant.

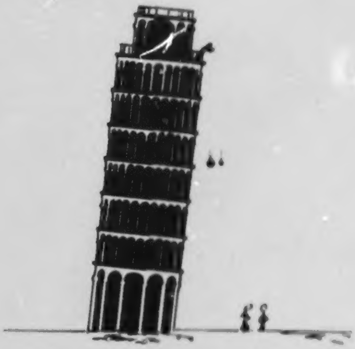


Figure 2

This simple experiment upset many people, for almost 2000 years earlier (about 350 B. C.), the great Aristotle had argued that the heavier the object the faster the fall. During the many years between, no one had really tested Aristotle's statement.

Returning to the experiment in Pisa, let's see what it is that Galileo discovered. Since the two stones were always side by side as they fell, we say that for any given number of seconds the two weights will travel the same distance.

What Galileo discovered was that the distance traveled by a falling object does not depend on its weight, but only depends on the length of time during which it has been falling. It is more common now to refer to falling bodies, rather than falling objects, but the meaning is, of course, the same. In present day language we would say that the distance traveled by a falling body is a function of the time spent in falling.

Galileo wanted to describe this relationship using the language of mathematics; he wanted a mathematical model for what physically happened.

Thus a rule or formula for finding the distance traveled will express the distance in terms of the time and will not involve the weight. This relationship, which we will not attempt to justify here, turns out to be:

> The number of feet traveled by a falling object is approximately equal to the product of 16 and the square of the number of seconds the object was falling.

Using a new symbol, $\approx$, to mean "approximately equal to", we shorten this statement to:

$$d \approx 16 \cdot t \cdot t,$$

or the more usual notation,

$$d \approx 16 \cdot t^2$$

where d is measured in feet, and t is measured in seconds.

For convenience, we often write this statement as follows:

$$d = 16t^2.$$

We should keep in mind that the answers are not exact in the physical sense, for not only is accurate, split-second timing impossible, but the "16" is an approximation that has been rounded off to the nearest whole number. (The meaning of the "16", in the statement, will be discussed in later courses.)

Let us consider an example of how this formula might be used.

Example: Suppose that two boys wished to measure the height of a waterfall, using the formula, $d = 16t^2$, and timing the fall with a stop watch. Actually, physical conditions, such as whether the cliff is vertical or slants outward at the bottom, and whether the spray of the water blocks vision, often make such experiments difficult to perform. Let us assume, however, that the very best conditions exist in this situation.

One boy went to the top of the falls, and the other stayed at the bottom of the falls with a stop watch. The boy on the cliff dropped a rock to the ground below. The other boy started the stop watch when the stone was dropped and stopped it when the stone hit the ground. He found that 3.5 seconds had passed. Now the boys made their calculations. They started with the formula that tells them that the height (in feet) of the falls can be estimated by the product of 16 and the square of the time (in seconds). This is how they made their calculations.

In this formula

$$d = 16t^2,$$

they replaced t by 3.5 to obtain

$$\begin{aligned} d &= 16(3.5)^2 \\ &= 16(3.5)(3.5) \\ &= 16(12.25) \\ &= 196 \end{aligned}$$

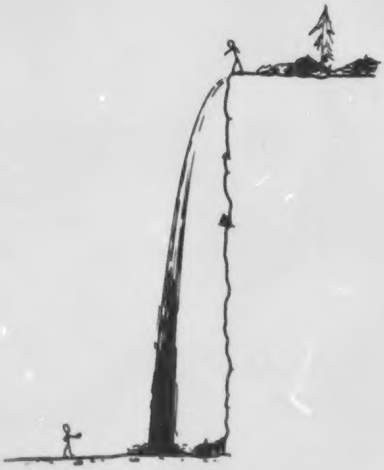


Figure 3

So the boys concluded that the falls were almost 200 feet high. Of course, it would be impossible for these boys to find the exact time difference using the stop watch. Although a stop watch measures time to the tenth of a second, for this experiment a mistake of a fraction of a second in measuring the time would produce quite different results, since the stone drops 112 feet between the third and fourth seconds!

In the following exercises you will find out a little more information about the distance traveled by the rock used in measuring the approximate height of the falls mentioned in the example above.

Exercises 2-2

1. Using the formula $d = 16t^2$, find the approximate distance traveled by the falling stone during each half-second during its fall. Fill in the missing values in the following table:

t : number of seconds

If t =	0	0.5	1	1.5	2	2.5	3	3.5
then d =	0	4	?	36	?	100	?	196

d : number of feet rock has fallen

2. Now find out how far the rock falls each half-second. Using the information of the table in Exercise 1, complete the following table.

During the following n half-second intervals	Number of feet rock fell
$t = 0$ to $t = .5$	4
$t = .5$ to $t = 1$	?
$t = 1$ to $t = 1.5$	?
$t = 1.5$ to $t = 2.0$	?
$t = 2.0$ to $t = 2.5$	?
$t = 2.5$ to $t = 3.0$	?
$t = 3.0$ to $t = 3.5$	?

3. Suppose the cliff had been somewhat higher; from your table in Exercise 2, can you conjecture (this means "make an intelligent guess") how many feet the rock would drop between $t = 3.5$ and $t = 4$? between $t = 4$ and $t = 4.5$? between $t = 4.5$ and $t = 5.0$? (Do not use the formula $d = 16t^2$. This problem needs very little arithmetic.)

In using the rule, or formula,

$$d = 16t^2,$$

we substitute or "input" a number as a value for t and then calculate or "output" the numerical value for d. For each input there is a corresponding output. This correspondence between the inputs and the outputs is an example of a "function". The formula $d = 16t^2$ serves to specify the function, but the formula itself is not considered to be the function. Thus

$$y = 16x^2$$

can be considered as designating the same function as $d = 16t^2$ does, because it produces the same correspondence between input and output values.

During this discussion you may have doubted that a feather would really fall as fast as a stone. Galileo was well aware that the feather falls more slowly, and he assumed that the air resistance was very great with such a light object. Later experiments showed this to be true. When a feather and a lead ball are dropped in a closed container from which all the air has been pumped they fall side by side and strike the bottom at the same time. The answers obtained using the formula $d = 16t^2$ are regarded as the correct answers in terms of the model, and are often good estimates of the real physical problem. For solid objects, such as lead balls, falling through short distances, the model is in rather close agreement with real life.

2-3. Some Examples of Functions

Functions occur everywhere in mathematics as well as in situations which may not be strictly mathematical. Because they occur so often we shall have many occasions to discuss them. We cannot hope to give a list of examples which will suggest all the possibilities. However, we shall give a few examples to suggest the wide variety of places in which the function idea is used. Let's begin by summarizing the information from the last section.

In the last section it was mentioned that the distance traveled by a falling body is a function of the time spent in falling. Here, it was noted that by substituting or "inputting" values for t (in seconds), we can calculate or "output" values for d (in feet). For each input, there is a corresponding output, or image. This correspondence between the acceptable inputs and the outputs is called a function. The mathematical sentence, $d = 16t^2$, helps to describe how the correspondence is to be made, and is only part of the description of the function. Other parts of the description of the function include the set of inputs and the set of outputs.

It sometimes helps our understanding to think of a function as a kind of machine (see Figure 4). This machine is equipped with a hopper and a spigot. When we place an input into the hopper the machine produces the corresponding output at the spigot.

Thus, for our falling body function, the output, or image, will always be 16 times the square of the input.

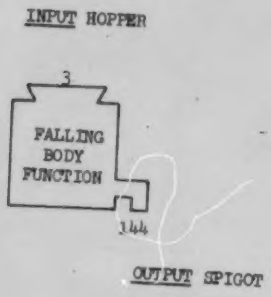

INPUT HOPPER
3
FALLING BODY FUNCTION
144
OUTPUT SPIGOT

Figure 4

In this example, we input a number of seconds and output a distance that corresponds to the number of seconds. Thus, with the inputs 1, 2, 3, 4, and 5, we get the outputs 16, 64, 144, 256, and 400. This can be shown as follows:

$$1 \longrightarrow 16, \qquad 2 \longrightarrow 64, \qquad 3 \longrightarrow 144,$$
$$4 \longrightarrow 256, \qquad 5 \longrightarrow 400, \qquad \underline{\text{and so on}}.$$

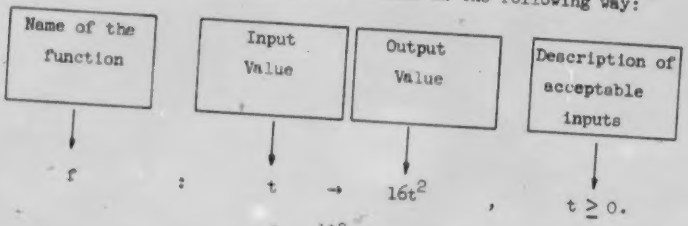
In general, an input of t (seconds) outputs $16t^2$ (feet), and diagrammatically, this is indicated in a fashion similar to the above:

$$f : t \rightarrow 16t^2.$$

The letter f is a name for the function. The remainder of the notation states that under this function, $16t^2$ corresponds to t. In this case, f names a function whose input is a number and whose output is a number. Since the values of t (time) are never negative in this function, the condition $t \geq 0$ (t is greater than or equal to 0) can be included in the description. Thus, we can now write:

$$f : t \rightarrow 16t^2; \; t \geq 0.$$

Schematically, we can think of this statement in the following way:


Name of the function
Input Value
Output Value
Description of acceptable inputs
f
:
t
→
16t²
,
t ≥ 0.

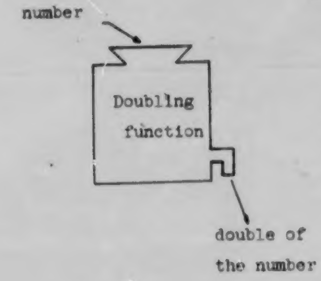
This way of representing a function is called arrow notation, and it can be read as follows:

(1) "the function f such that to every t there corresponds $16t^2$, and t is a non-negative number", or

(2) "the function f such that t is the input and $16t^2$ is the output, and t is a non-negative number", or simply

(3) "f takes t into $16t^2$, where $t \geq 0$".

Two other examples of functions, which can be illustrated by a machine, are the doubling function and the squaring function.



For example:

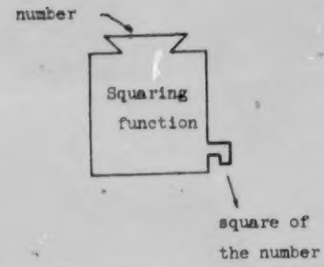
1 has the image 2, or 1 → 2.

2 has the image 4, or 2 → 4.

3 has the image 6, or 3 → 6.

4 has the image 8, or 4 → 8.

The arrow notation for this function is $f : n \rightarrow 2n$.



For example:

1 has the image 1, or 1 → 1.

2 has the image 4, or 2 → 4.

3 has the image 9, or 3 → 9.

4 has the image 16, or 4 → 16.

The arrow notation for this function is $f : n \rightarrow n^2$.

Suppose you say, "The time I wake up in the morning is a function of the time I go to bed, and I always sleep 7 hours." If t represents the time you go to bed, then t + 7 is the time you wake up. We can describe this function in arrow notation by

$$f : t \rightarrow t + 7.$$

The f, used here, is probably the most common name for a function, but F, g, h, and other letters are also used as names for functions. We could just as well describe this function by using these other letters, for example,

$$F : t \to t + 7$$

$$g : t \to t + 7$$

$$h : t \to t + 7$$

Functions may establish correspondences between objects other than numbers.

Example 1:

Names of Four Buildings in New York City	Height (in feet)
Empire State Building	1472
Chrysler Building	1046
RCA Rockefeller Center	850
Pan Am Building	808

We have four inputs and the corresponding output for each. In this case, the height (output) is a function of the building (input).

Example 2:

The world's tallest known trees are all in California. If we input the name of one of these big redwood giants, we output the height; for example, the height is a function of the name of the tree.

Names of Six Redwood Trees	Height (in feet)
Howard Libbey Tree	367.6
Harry W. Cole Tree	365.4
National Geographic Tree	364.5
Rockefeller Tree	356.5
Founders Tree	352.6
Redwood Creek Grove	352.3

It is also true that if we input the heights of these particular trees we would output the name of the tree. In this case, the name of the tree is a function of the height.

It is important to note that when we reverse the direction of a correspondence which is a function, the resulting correspondence is not always a function.

Example 1:

Below is a table which shows a correspondence between pairs of numbers, a correspondence from the top row of numbers to the bottom row.

Input	0	1	2	3	4	5	6
Output	$12\frac{1}{2}$	8	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$

You can see that corresponding to each input there is exactly one output. Therefore, this correspondence is a function.

If the direction of the above correspondence is reversed, then the table below shows the following correspondence between pairs of numbers.

Input	$12\frac{1}{2}$	8	$4\frac{1}{2}$	2	$\frac{1}{2}$	0	$\frac{1}{2}$
Output	0	1	2	3	4	5	6

You can see that there are two outputs, 4 and 6, corresponding to the input $\frac{1}{2}$. Therefore, this second correspondence is not a function because there is more than one output for a particular input.

Example 2:

Every article in your neighborhood hardware store has a price. The price is a kind of label associated with the article. Thus, the price is a function of the article.

We may describe the correspondence by such a statement as, "Every article in this store has a price." Or, we might say, "Associate with every article in the store its price." Since, for each article there is one price, we have a functional relationship. We can use the arrow notation to describe this function, where price is a function of the article,

p : article → price.

Now, suppose we say, "Associate with every price (for which the store sells goods) an article." Since, for some prices, there would be not one, but several articles, we do not have a functional relationship.

Exercises 2-3a

(Class Discussion)

1. In some cases below, the statement describes a function. In other cases, the statement does not describe a function. If the statement describes a function use the arrow notation to represent the function. If the statement does not describe a function explain why it does not.

   (a) Associate with a point in space a line through the point.

   (b) Associate with two different points in space a line through these points.

   (c) Associate with a point in space a plane through the point.

   (d) Associate with two different points in space a plane through these points.

   (e) Associate with three non-collinear points in space a plane through these points.

Exercises 2-3b

Below are four statements which describe correspondences between pairs of things. Write each statement on a sheet of paper, and, under each of them, write a new statement which shows that the first correspondence has been reversed. You should have 8 statements (4 pairs) on your paper.

Now, follow these instructions:

(1) Decide whether the first statement, in each pair of statements, describes a function, and then

(2) decide whether the new statement, in each pair of statements, describes a function.

(3) If any of these 8 statements describe a function, then represent this function in terms of arrow notation.

1. (a) Associate with each member of your mathematics class his or her age in years.

   (b) Associate with each state of the United States the name of its present governor.

(c) Associate with each height, in inches, a citizen of the United States.

(d) Associate with a first name for a person, a last name.

2-4. Ways of Representing Functions

We have seen that functions may be represented by using the arrow notation. Since a function is a correspondence it is reasonable to expect that a function may be represented in many other ways. In this section, we will discuss some of the more common ways of expressing functions.

(a) A function may be represented by a statement. For example:

With two distinct points in the plane associate the distance between them.

This may be expressed by using the arrow notation:

$$f : (A,B) \rightarrow \text{distance between } A \text{ and } B,\ A \neq B.$$

Here, inputs are pairs of points in the plane, while outputs are positive numbers.

(b) A function may be expressed by a table. For example:

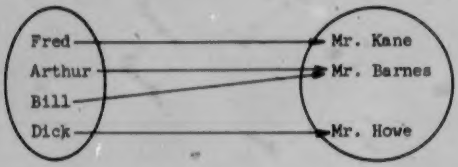
In a basketball game, the number of points scored by each player on the Lincoln High School team is shown in the following table.

Player	Points
Andrews	12
Brown	8
Davis	9
Taylor	10
Harris	6
Faber	10
Elliot	7

In this case, we have a function from a set of players to a set of positive numbers. The table actually states the correspondence completely and is probably the most useful form of representing the function. However, we may indicate the function by using the arrow notation, as follows:

h : player → number of points scored by player.

(c) A function may be indicated by a diagram which pictures the correspondence. For example:



This diagram shows the correspondence between a set of boys and the set of their fathers. Although we could do so, it would be of little use to express this function by using the arrow notation.

(d) A function may be described by a formula. Think of a formula as a rule which has been abbreviated and written in the form of a mathematical sentence. For example:

$$P = 4s.$$

In this formula, s (representing inputs) stands for the length of a side of a square and P (representing outputs) stands for the perimeter of the square (see Figure 5).

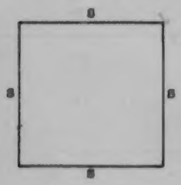


Figure 5

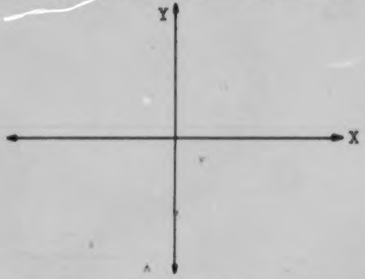
In describing a function by a formula, such as $P = 4s$, the letter representing the output occurs only once in the formula and is alone on one side of the equality sign. If this function is named g, then it can be represented in arrow notation as:

$$g : s \rightarrow 4s,\ s > 0.$$

Remember: $s > 0$ means that "s is greater than 0".

(e) A graph is one of the most useful ways of representing a function. Before we learn to draw the graph of a function, let's agree to the following statements:

(1) "To draw the graph of a function" means to find a collection of points in the coordinate plane



such that they will accurately picture the certain kind of association or correspondence which we call a function.

(2) Inputs of a function will be associated with the coordinates of points on the X-axis (the horizontal number line).

(3) Outputs of a function will be associated with the coordinates of points on the Y-axis (the vertical number line).

Now, we will draw the graph of the function, h, described by

$$h : x \rightarrow 2x,\ x \geq 0.$$

Remember: $x \geq 0$ means that "x is greater than or is equal to 0".

In order to draw the graph we will first choose a few inputs, represented by x, and calculate the corresponding outputs, represented by 2x. Our results are shown below.

Input: x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4
Output: 2x	0	1	2	3	4	5	6	7	8

Notice that inputs were chosen so that the calculation of outputs was easy work!

Notice also that an increase in the input causes an increase in the output.

Now, we plot the points corresponding to these pairs of numbers: $(0,0)$, $(\frac{1}{2},1)$, $(1,2)$, $(\frac{3}{2},3)$, $(2,4)$, $(\frac{5}{2},5)$, $(3,6)$, $(\frac{7}{2},7)$, $(4,8)$.

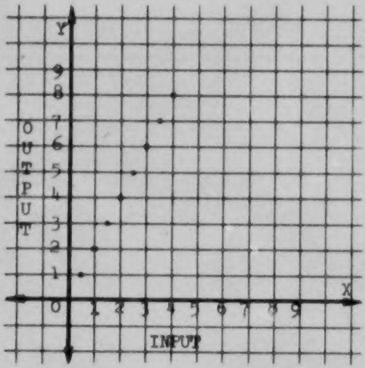


Figure 6a

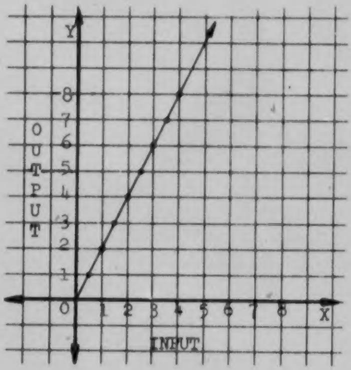


Figure 6b

Figure 6a shows these points. Of course, the graph of this function consists of more points than are shown. You can see that it would not be possible to make a complete table for the function. Therefore, it would also be impossible to plot all the points belonging to its graph. However, as shown in Figure 6b, the points we have plotted seem to be collinear. In fact, if we find other number pairs which belong to this function, such as $(\frac{1}{4}, \frac{1}{2})$, $(\frac{9}{2}, 9)$, $(5,10)$, their corresponding points seem to be collinear with those already plotted. Let's agree that the graph of this function is a ray, whose endpoint is the origin, and which passes through the first quadrant as shown in Figure 6b. Notice that this graph is rising from left to right. This should not surprise us because we have already noted that an increase in the input causes an increase in the output. A function whose graph has this characteristic is often called an increasing function.

To gain more experience in drawing graphs of functions, we will draw the graph of the function, f, described by

$$f : x \rightarrow x^2, \ 0 \leq x \leq 3.$$

Remember: $0 \leq x \leq 3$ means that "$x \geq 0$ and $x \leq 3$".

Again, in order to draw the graph we will first choose a few inputs, represented by x, and calculate the corresponding outputs, represented by x^2. Our results are shown below.

Input: x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Output: x^2	0	$\frac{1}{4}$	1	$\frac{9}{4}$	4	$\frac{25}{4}$	9

Again, notice 2 things:

(1) inputs were chosen so that the calculation of outputs was easy work;

(2) an increase in the input causes an increase in the output.

Now, we will plot the points corresponding to these number pairs: $(0,0)$, $(\frac{1}{2}, \frac{1}{4})$, $(1,1)$, $(\frac{3}{2}, \frac{9}{4})$, $(2,4)$, $(\frac{5}{2}, \frac{25}{4})$, $(3,9)$.

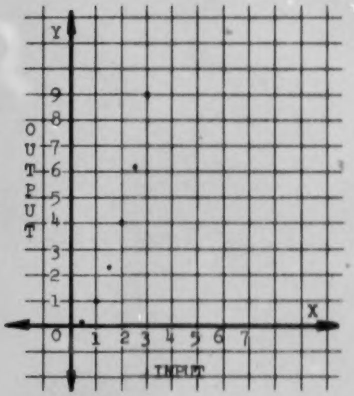


Figure 7a

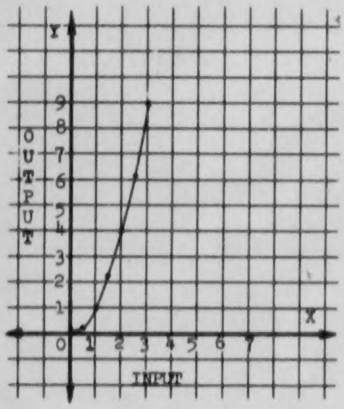


Figure 7b

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Figure 7a shows these points. Of course, the graph of this function also consists of more points than are shown. You can see that it would not be possible to make a complete table for this function either. Therefore, it would, again, also be impossible to plot all the points belonging to its graph. However, it is easily seen that the points we have plotted are not collinear. But, by drawing a smooth curve through the plotted points, as shown in Figure 7b, we can obtain a fairly accurate sketch of the true graph of the function. Notice that this graph, too, is rising from left to right. Recall that an increase in the input causes an increase in the output. Therefore, we should expect the graph of this function to behave as it does (rise from left to right). Thus, this function, too, can be called an increasing function.

When drawing the graph of a function, we may choose any unit of measure for the X-axis and any unit of measure for the Y-axis. This information will be useful when drawing the graphs of certain functions.

We have learned to draw the graph of a function by making a table of inputs and outputs. Now, let's learn to read the graph of a function (see Figure 8).

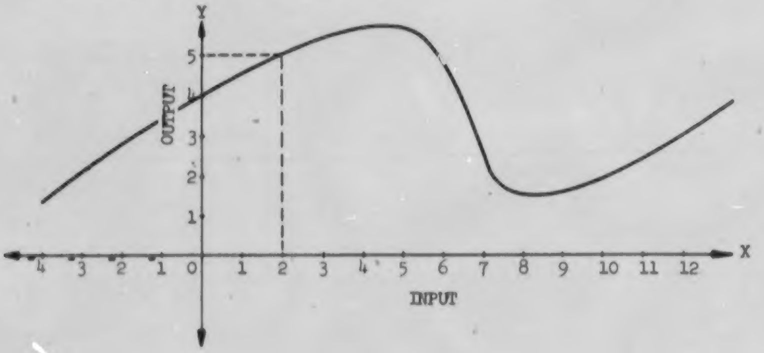


Figure 8

We can read this graph in the following way. Suppose we wish to know the output for the input 2. We first locate the point labeled 2 on the input axis (here the horizontal axis). Then we search along the vertical line through this point until we find a point on the graph. The horizontal line through this point of the graph passes through 5 on the output axis. Hence, the point with coordinates (2,5) is a point on the graph, and 5 is the output corresponding to the input 2. We can see that by using this method, a single output can be found for each input. The reason is that each vertical line intersects the graph in just one point.

It would be difficult to express this function by using the arrow notation. In some cases, a graph is the only convenient way of representing a function.

In Figures 9a and 9b, we see graphs which do not represent functions.

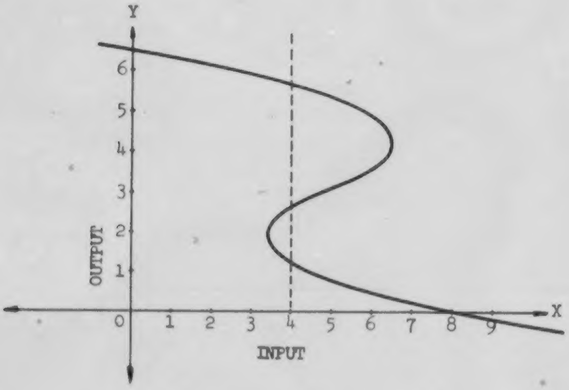


Figure 9a

In Figure 9a, what would be the output corresponding to an input of 4? Here, we see that instead of getting a single definite image, three possible images are put forth. Thus, there is no one output corresponding to each input. So, although this graph may be useful for other purposes, it does not represent a function.

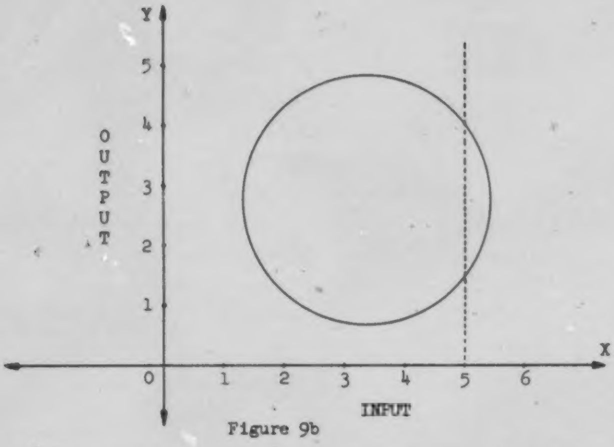


Figure 9b

In Figure 9b, what would be the output corresponding to an input of 5? Here, we see that this input will be associated with two images. Again, there is no one output corresponding to each input. Therefore, this graph does not represent a function either.

Check Your Reading

1. Can you name 5 ways of representing a function?
2. What does "to draw the graph of a function" mean?
3. Which axis in the coordinate plane is associated with the set of inputs of a function?
4. Which axis in the coordinate plane is associated with the set of outputs of a function?
5. What is a function called if its graph rises from left to right?
6. What does "to read the graph of a function" mean?
7. When you look at a graph, how can you decide whether it represents a function?

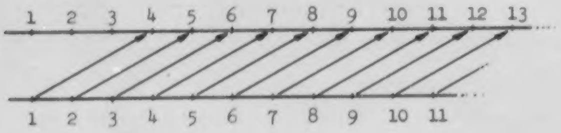
Exercises 2-4

1. Associate with each counting number, c, its triple, $3c$.

   (a) Express this function by using the arrow notation.

   (b) Is it practical to express this function by means of a table? Give a reason for your answer.

2. The following diagram represents a function.



   (a) What is the input if the output is 6?

   (b) What is the output if the input is 2?

   (c) What is the output if the input is 10?

   (d) Express this function by using the arrow notation.

3. Study the arrow notation describing the function n.

   $n : w \rightarrow 2w + 1$, w represents a whole number.

   (a) What is the output for an input of 4?

   (b) What is the input for an output of 15?

   (c) Can an output be an even number? Give a reason for your answer.

4. A car travels at a steady rate of 40 miles per hour. Associate with the time (t) in hours the distance (d) in miles covered by the car.

   (a) Express this function by a formula.

   (b) Express this function by using the arrow notation.

   (c) What is the output if the input is $3\frac{1}{2}$?

   (d) What is the input if the output is 150?

5. The function, g, associates with each positive number s the area of a square whose side is s units in length.

$$g : s \rightarrow s^2,\ s > 0.$$

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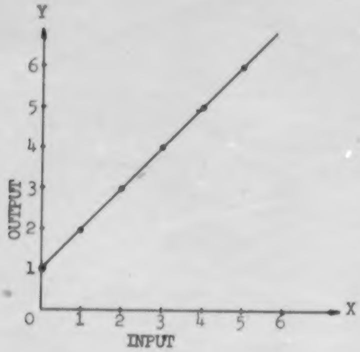
(a) Find the outputs for inputs of 3 and 6. How are these outputs related?

(b) Find the outputs for inputs of 4 and 8. How are these outputs related?

(c) Does the same relationship between inputs and outputs hold for any pair of inputs such that one is double the other?

6. The graph below represents a function.



(a) What is the output for an input of 2?

(b) What is the output for an input of 5?

(c) What is the input for an output of 4?

(d) Does this graph represent an increasing function? Explain.

7. Draw graphs of the following functions.

(a) $k : x \rightarrow 2$.

(b) $\ell : x \rightarrow {}^-1$.

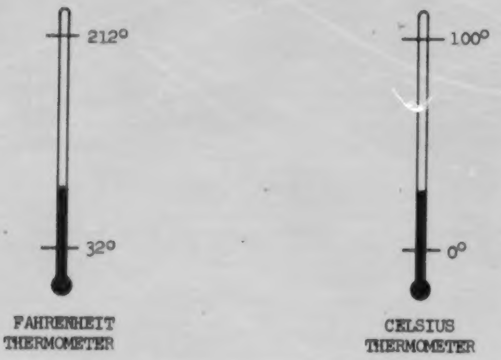
Use the language of geometry to accurately describe these graphs.

Explain why functions such as k and ℓ are often called constant functions.

8. In measuring temperature we ordinarily use the Fahrenheit scale. However, in many parts of the world and also in the science laboratory, the Celsius scale (formerly called Centigrade scale) is used. The

relationship between the two scales is a function. The function may be expressed by the following formula.

$$F = \frac{9}{5}C + 32$$



(a) What is the output for an input (C) of 0? What is the scientific significance of this output?

(b) What is the output for an input (C) of 100? What is the scientific significance of this output?

(c) Complete the given table:

Input: C	0	10	40	60	80	100
Output: $\frac{9}{5}C + 32$	?	?	?	?	?	?

(d) Choose convenient units of measure for both axes, and draw the graph of this function.

9. A certain amount of a gas shut up in a container exerts pressure on the walls. If the size of the container can be changed, as by a piston, then the volume (V) of the container determines the pressure, in agreement with the function

$$g : V \rightarrow \frac{50}{V}, \ V > 0.$$

(a) Complete the given table:

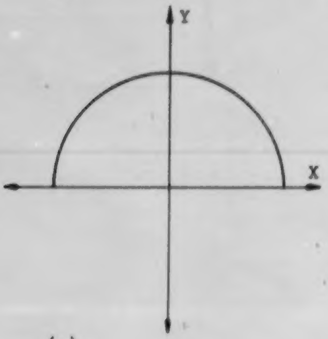
Input: V	1	2	4	5	10	20	25	50
Output: $\frac{50}{V}$	?	?	?	?	?	?	?	?

(b) Draw the graph of this function.

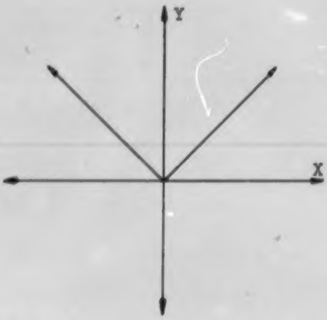
(c) Is the function, g, an increasing function? Explain.

10. Which of the following graphs represent functions?

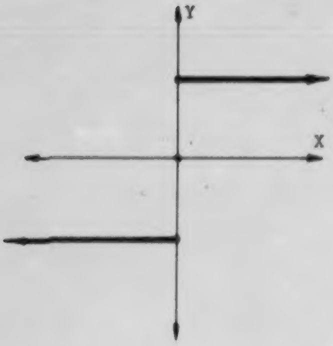
(a)



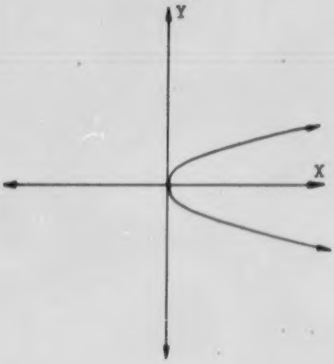
(b)



(c)



(d)



2-5. Discovering the Usefulness of Graphs

We have seen that graphs can be used to give clear pictures of different kinds of functions. More than just a picture of some relationship, a graph can show other important patterns of behavior of a function on closer examination. Consider the following example of charting growth in height.

Example: Tom Jones' dad liked to keep records of facts and figures. When Tom was born and on each of his successive birthdays, Mr. Jones wrote down Tom's height in inches. He stopped keeping records when Tom was 21 years old. Here are his records.

Age	0	1	2	3	4	5	6	7
Height	21	$25\frac{1}{4}$	$28\frac{3}{4}$	$31\frac{1}{4}$	$33\frac{1}{2}$	$35\frac{1}{2}$	$37\frac{1}{4}$	$39\frac{1}{4}$

Age	8	9	10	11	12	13	14
Height	41	44	$46\frac{1}{2}$	$48\frac{3}{4}$	51	$52\frac{3}{4}$	$54\frac{1}{4}$

Age	15	16	17	18	19	20	21
Height	56	$58\frac{1}{4}$	64	71	71	71	71

In some of our other examples, the tables don't completely represent the function. If the number of acceptable inputs is not too many, then by showing all these inputs together with their corresponding outputs, a complete representation of the function can be given. Thus, the information that we have above, in table form, on Tom's height at each birthday through his twenty-first, represents the entire function: namely, Tom's height-at-birthday function. If we graph the function, Tom's growth pattern shows up more clearly.

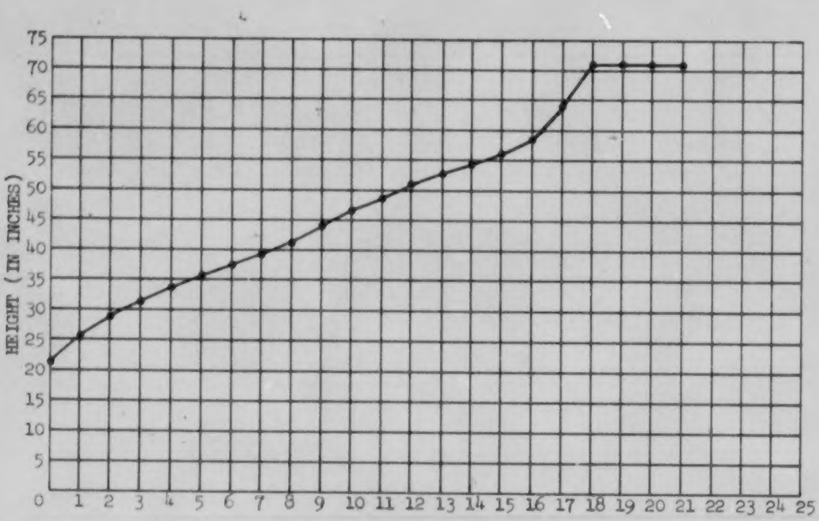


Figure 10

There are two things to notice about this graph. First, we have used different scales on the two axes for convenience in representing the facts. Second, although only the points plotted belong to the graph of the function, we have joined consecutive points by segments to make the graph easier to read.

We cannot read the graph as well as we can read the table. However, we can find a number of properties of the height-at-birthday function from the graph more quickly than we can from the table. We will discuss a number of these properties.

Before doing so, let us agree to use the symbol h (suggesting height) as the name for this function. Throughout this section, h represents Tom's height-at-birthday function:

h : age → height at birthday for the age.

Exercises 2-5a

(Class Discussion)

Look at Figure 10.

1. (a) Is the graph of the function h rising or falling, from left to right, for the inputs (ages) from 0 to 18? Explain.

   (b) For these inputs, is function h an increasing or a decreasing function?

   (c) What meaning can we give to the answer to question 1(a)?

2. (a) What is happening to the graph of the function h for the inputs (ages) from 18 to 21? Explain.

   (b) What meaning can we give to the answer to question 2(a)?

   (c) What word do you think we can use to describe the behavior of h for the inputs (ages) from 18 to 21?

3. The graph of the function h consists of 22 points. Perhaps we can discover some new information by examining the segments joining any two consecutive points belonging to this graph.

   (a) Is there a segment which seems to be steeper than any other segment?

   (b) What meaning can we give to the answer to question 3(a) if we want to relate it to Tom's growth pattern?

   (c) Can we find the coordinates for both of the endpoints belonging to this "steepest" segment by looking at the table of Tom's age and height?

   (d) What are the coordinates of the left endpoint and the right endpoint of this segment?

   (e) Between which two birthdays did Tom have his greatest growth?

   (f) What arithmetic problem can we do to find Tom's growth between these two birthdays? How much did Tom grow during this 1 year period?

   (g) Can we tell from looking at the graph of h how Tom's growth between his 16th and 17th birthdays compares with any other 1 year period?

(h) What arithmetic problem can we do to find Tom's growth between his 16th and 17th birthdays? How much did Tom grow between these birthdays?

The class discussion exercises suggest that we consider how much Tom grew from one birthday to the next. The change in his height during each year may be associated with his age at the end of the year. In this way we describe a new function. In Exercise 3(f) we found that Tom grew 7 inches between his 17th and 18th birthdays; thus his growth during his 18th year was 7 inches. We make 7 correspond to 18 under the "growth function". If we represent the growth function by g (suggested growth), we have

g : age → growth between preceding birthday and birthday for the age.

The growth is, of course, the change in height. Thus the output under g is given by

(height at birthday of given age) minus (height at preceding birthday).

Also, in Exercise 3(h), we found that when 17 is an input for the growth function g, then the output is $5\frac{3}{4}$.

The graph of the function h (see Figure 10) rises rather steadily between Tom's second birthday and his sixteenth birthday. This represents the fact that Tom grew at a fairly steady rate during this period. In terms of the growth function g, this statement means that the outputs for g are nearly the same for all these inputs. Let us check on this by studying the pattern of Tom's growth in more detail. We may do this by going to the table for the function h and using the facts given there to find outputs under the growth function.

We now give part of the table for the growth function g. Then we will use this table to answer different questions about Tom's growth.

Input for growth function g	Growth from age	to age	Output for growth function g
3	2	3	$31\frac{1}{4} - 28\frac{3}{4} = 30\frac{5}{4} - 28\frac{3}{4} = 2\frac{2}{4} = 2\frac{1}{2}$
4	3	4	$33\frac{1}{2} - 31\frac{1}{4} = 33\frac{2}{4} - 31\frac{1}{4} = 2\frac{1}{4}$
5	4	5	$35\frac{1}{2} - 33\frac{1}{2} = 2$
6	5	6	$37\frac{1}{4} - 35\frac{1}{2} = 36\frac{5}{4} - 35\frac{2}{4} = 1\frac{3}{4}$
7	6	7	$39\frac{1}{4} - 37\frac{1}{4} = 2$
8	7	8	$41 - 39\frac{1}{4} = 40\frac{4}{4} - 39\frac{1}{4} = 1\frac{3}{4}$
9	8	9	$44 - 41 = 3$
10	9	10	$46\frac{1}{2} - 44 = 2\frac{1}{2}$
11	10	11	$48\frac{3}{4} - 46\frac{1}{2} = 48\frac{3}{4} - 46\frac{2}{4} = 2\frac{1}{4}$
12	11	12	$51 - 48\frac{3}{4} = 50\frac{4}{4} - 48\frac{3}{4} = 2\frac{1}{4}$

(1) During the ten-year period from age 2 to age 12, what was Tom's largest yearly growth? When did this occur? By examination of the outputs shown in the table we find the largest output is 3. So Tom's greatest yearly growth during the ten-year period was 3 inches, and this happened between the ages of 8 and 9.

(2) During the ten-year period, what was Tom's smallest yearly growth? When did this occur? The smallest output shown in the table is $1\frac{3}{4}$. Do you find that there were two years during which he made the least growth? The first was between his fifth and sixth birthdays, and the second was between his seventh and eighth birthdays.

Now consider another question concerning the ten-year period between Tom's second and twelfth birthdays. What was Tom's *average rate of growth* during this time? Another way of putting this question is: "Suppose that Tom had grown the *same amount* each year from the age of 2 to the age of 12. What would this amount have had to be so as to produce the same total growth over the ten-year period?" We can answer this question by finding the change in Tom's height between the ages of 2 and 12, and dividing this total growth into 10 equal parts.

Height at age 12 was 51 inches.

Height at age 2 was $28\frac{3}{4}$ inches.

Total growth from age 2 to age 12 was $22\frac{1}{4}$ inches

because $51 - 28\frac{3}{4} = 50\frac{4}{4} - 28\frac{3}{4} = 22\frac{1}{4}$.

$$\text{Average growth per (for each) year} = \frac{\text{Total growth}}{\text{Number of years}}$$

$$= \frac{22\frac{1}{4}}{10}.$$

We can simplify this fraction by multiplying by the number 1 (named as $\frac{4}{4}$):

$$\frac{22\frac{1}{4}}{10} = \frac{4}{4} \cdot \frac{22\frac{1}{4}}{10} = \frac{4 \cdot 22\frac{1}{4}}{4 \cdot 10} = \frac{88 + 1}{40} = \frac{89}{40} = 2\frac{9}{40}.$$

If we remember that the numerals "$\frac{1}{4}$" and ".25" are just different names for the same number, then we could simplify this fraction by replacing "$22\frac{1}{4}$" by "22.25":

$$\frac{22\frac{1}{4}}{10} = \frac{22.25}{10} = 2.225.$$

In any case, we see the *average rate of growth* is just about $2\frac{1}{4}$ inches. *Remember*: $2\frac{9}{40} = 2.225$ and *either* numeral names a number a little less than $2\frac{1}{4}$.

Exercises 2-5b

For these exercises, refer to the table and graph of Tom's height-at-birthday function h (see pages 81-82).

1. We know from the class discussion exercises that Tom grew more between his 17th and 18th birthdays than during any other recorded 1 year period. We know also that Tom had his next greatest growth between his 16th and 17th birthdays. Look at the graph of h and decide when Tom had his third greatest yearly growth. Give the ages between which this growth happened and then use the table to find how much Tom grew.

2. Look at the graph of h and decide when Tom had his least yearly growth. Give the ages between which this growth happened and then use the table to find how much Tom grew.

3. (a) From the graph find at which birthday Tom's height first became greater than twice his height at birth. When did his height first become greater than three times his height at birth?

   (b) Tom's final height was how many times his height at birth? Name this number in three different ways: by a fraction, by a mixed numeral, and by a decimal numeral (rounded to the nearest hundredth).

4. (a) Make a complete table showing Tom's growth function g. Corresponding to the input 1 give the growth from age 0 to age 1. We have started the table for you. Don't forget: outputs corresponding to inputs from 3 to 12 are shown on page 85.

Table: Growth function g

Age	1	2	3	...
Growth since preceding birthday	$4\frac{1}{4}$	?	$2\frac{1}{2}$	

   (b) Why is 0 an acceptable input for the function h, but not an acceptable input for the function g?

   (c) The largest input for the function h is 21. Is it also an acceptable input for our function g?

5. (a) Draw the graph of Tom's growth function g from problem 4. Choose a scale on the vertical axis so that your graph will be about 3 inches high.

(b) Join consecutive points, belonging to the graph of g, by segments. Do these segments belong to the graph of the growth function?

(c) Compare the graphs of functions h and g for the inputs (ages) from 1 to 6. Notice that the graph of function h is increasing for these inputs while the graph of function g is decreasing. In terms of Tom's growth pattern, what does this mean?

(d) Compare the graphs of functions h and g for the inputs (ages) from 14 to 18. Notice that, for these inputs, the graphs of both functions are increasing. In terms of Tom's growth pattern, what does this mean?

6. What was Tom's average rate of growth during his growing period from birth to the age of 18 years?

Exercises 2-5c

The Evans family consists of Mr. and Mrs. Evans and three children. When the children were young Mr. and Mrs. Evans decided to start a savings fund to pay for the college education of their children. The correspondence below shows how much money was in the fund at the end of each year.

The number of years the Evans have had a savings fund	The number of dollars in the savings fund
0	$ 0
1	1,020
2	1,680
3	3,312
4	2,906
5	4,850
6	6,784
7	8,120
8	6,215
9	7,028
10	9,560
11	8,420
12	10,164

1. This correspondence between number of years and number of dollars represents a function.

   (a) How do we know that the numbers in the left-hand column are the inputs for this function?

   (b) What name can we give to the numbers in the right-hand column?

2. The graph of this function consists of a set of 13 points. With the exception of the point whose coordinates are (0,0),

   (a) in which quadrant of a coordinate plane can you draw the graph of this function?

   (b) What can be done to make the graph of this function easier to read?

3. Draw a graph of this function: Let points on the horizontal axis represent the years and points on the vertical axis represent the savings. (A convenient scale on the vertical axis is 1 unit of

measure = $500.) Join consecutive points of this graph by segments.

4. The segments, joining consecutive points of this graph, seem to have 2 different kinds of steepness.

Some segments are steep in the sense that they rise from left to right. For example,

Some segments are steep in the sense that they fall from left to right. For example,

Now, let's find meanings to assign to these segments, in terms of the Evans family and their savings fund. It seems reasonable to state that a rising segment should mean an increase in savings. What meaning should we give to a falling segment?

5. From the graph find the year in which there was the greatest increase in savings. How does the graph show this?

6. From the graph determine the year in which there was the sharpest drop in savings. How does the graph show this?

7. From the graph determine the longest period of years during which there were only increases in savings. How does the graph show this?

8. During the fourth year of savings the Evans family bought a new house. Explain how this information "fits" with the segment joining the points whose coordinates are (3, 3312) and (4, 2906).

9. During the tenth year of savings Mr. Evans received a big increase in salary. Explain how this information "fits" with the segment joining the points whose coordinates are (9, 7028) and (10, 9560).

10. (a) How much money did the Evans family save over a period of 12 years?

    (b) For the entire period what was the average change in savings per year?

2-6. The Identity Function

Certainly one of the simplest of all functions is the identity function:

$$I : x \to x.$$

If we imagine this function as a machine operation, then whatever number we use as input is returned to us as output. This happens, for example, when we put a coin into a machine for cold drinks, and the machine has run out of cold drinks; the coin is returned to us. In Figure 11a, the identity function machine outputs $^{-}17$ for the input $^{-}17$.

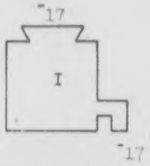


Figure 11a

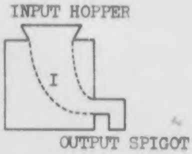


Figure 11b

It certainly ought to be easy to build such a machine. The inner workings would consist entirely of a tube connecting the input hopper and the output spigot, as shown in Figure 11b.

Graphing this function is, of course, quite simple. Take a particular input, say 3, and its corresponding output, also 3, and plot the point corresponding to (3,3). Similarly, the points corresponding to the number pairs (0,0), (1,1), (5,5), *and so on*, belong to this graph.

In connection with the identity function, let's examine the set of four prints: O with coordinates (0,0), B with coordinates (3,0), P with coordinates (3,3), and A with coordinates (0,3). It is clear that these points are the vertices of a square. The diagonal of this square (see Figure 12) from O(0,0) to P(3,3) divides this square into two triangles having the same shape and size, $\triangle$ AOP and $\triangle$ BOP.

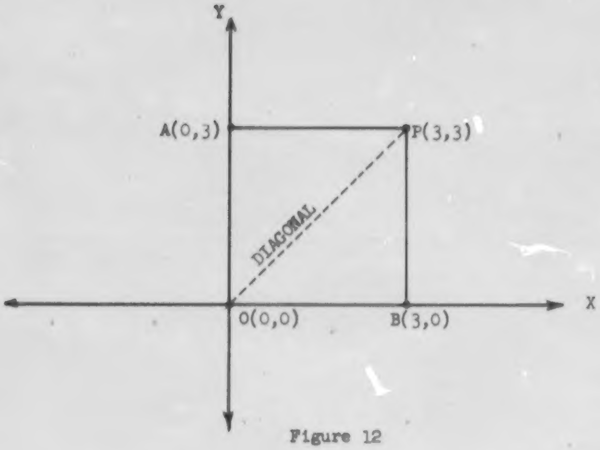


Figure 12

To understand this, think of cutting a square out of paper, as in Figure 13a, and folding the square over the diagonal $\overline{OP}$, as suggested in Figure 13b.

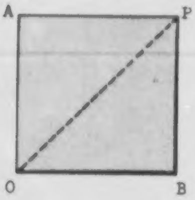


Figure 13a

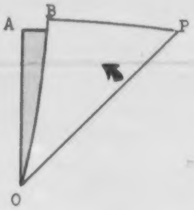


Figure 13b

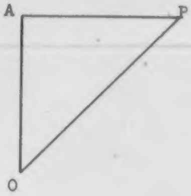


Figure 13c

After this folding, the points A and B will coincide. See Figure 13c. So will the angles: ∠AOP and ∠BOP. (You have seen this done many times in folding square napkins.) Thus the angles, ∠AOP and ∠BOP, have the same measure so that in Figure 13a the ray $\overrightarrow{OP}$ bisects the angle BOA formed by rays on the coordinate axes. The same reasoning holds for any points with the two coordinates equal such as (2,2), (4.5,4.5), (8,8), and so on.

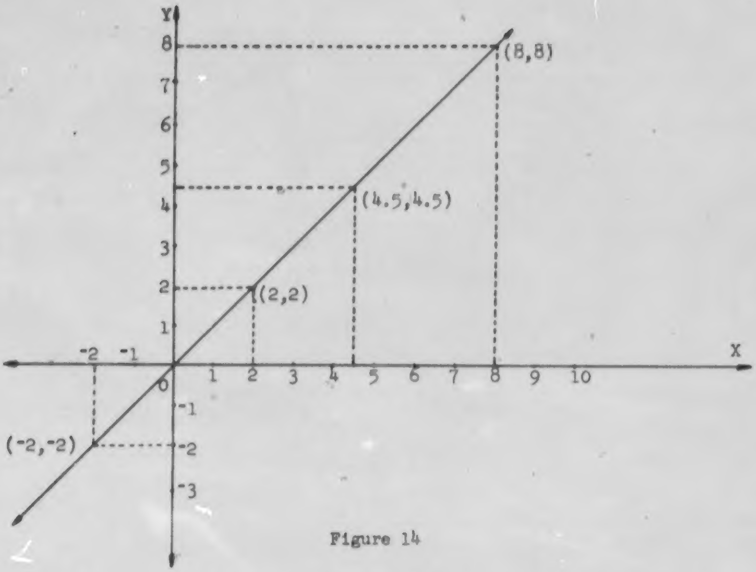


Figure 14

So all these points lie on the ray bisecting the first quadrant. If we extend this ray into the third quadrant, thus having an entire line, we see that all points on this line have both coordinates the same. Thus the graph of the identity function is a line through the diagonal of the square in Figure 12. The graph of this simple, but important, function is drawn for you in Figure 14. Study it carefully.

7

Exercises 2-6

1. On one coordinate plane, draw the graphs of the following functions.
   - (a) $I : x \rightarrow x$
   - (b) $h : x \rightarrow x + 2,\ x \geq 0$
   - (c) $k : x \rightarrow x + 3,\ x \geq 0$
   - (d) $\ell : x \rightarrow x + 5,\ x \geq 0$
2. We know that the graph of the identity function I can be described as a line.
   - (a) How can we describe the graphs of functions h, k, and ℓ?

(b) How are the graphs of h, k, and ℓ related to the graph of I?

(c) Are functions I, h, k, and ℓ increasing functions? Explain.

2-7. New Kinds of Functions

Example 1:

Mr. Gray works for the National Metals Corporation. He is often called upon to work overtime and has agreed to do so provided he is paid \$4 per hour and is to get a full hour's pay for any part of an hour that he works. For example, he is to be paid for 11 hours when he works $10\frac{1}{4}$ hours. This correspondence between number of hours worked and pay received is a function. The input is the number of hours worked and the output is the corresponding pay; and corresponding to each input there is exactly one output.

If we try to describe this function by using the arrow notation we run into difficulties. We may get a clearer understanding of the function by drawing a graph. When we consider the set of inputs and the set of outputs we note that they do not follow a familiar pattern. For example,

For all inputs, n,

if $0 < n \leq 1$, then the output is 4;

if $1 < n \leq 2$, then the output is 8;

if $2 < n \leq 3$, then the output is 12;

if $3 < n \leq 4$, then the output is 16;

if $4 < n \leq 5$, then the output is 20, and so on.

Let us draw a graph of these inputs and outputs, plotting the inputs on the horizontal axis and the outputs on the vertical axis. Note that we are using different scales on the horizontal and vertical axes.

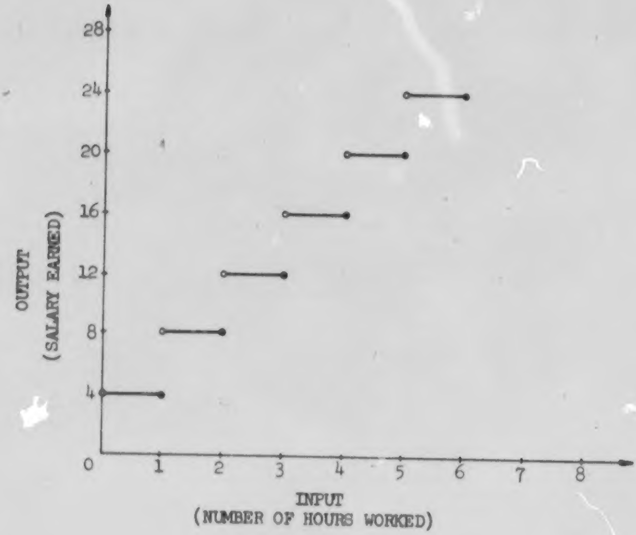


Figure 15

The tiny circle at the left of each horizontal segment shows us that this endpoint does not belong to the graph.

We see that the graph of this function is made up completely of a number of horizontal pieces. A function whose graph has this property is called a step function.

Example 2:

In the manufacture of certain machine parts at the National Metals Corporation it is necessary to cut 1-foot lengths of steel out of large sheets of steel of standard width. In this process, a leftover piece of steel with length less than 1 foot is considered waste. Thus, for a sheet of steel of standard width that is 6 feet 5 inches in length, six machine parts may be obtained. The extra 5 inches are waste. This correspondence between lengths of steel and the numbers of machine parts obtained is a function. The set of inputs consists of the various steel sheet lengths used and the set of outputs consists of the numbers of machine parts obtained.

Again, a graph may help us to get a clearer understanding of the function. The inputs and outputs do not follow a familiar pattern. For example,

For all inputs n,

if $0 \leq n < 1$, then the output is 0;

if $1 \leq n < 2$, then the output is 1;

if $2 \leq n < 3$, then the output is 2;

if $3 \leq n < 4$, then the output is 3;

if $4 \leq n < 5$, then the output is 4; and so on.

Let us draw a graph of this function, plotting the inputs on the horizontal axis and the outputs on the vertical axis.

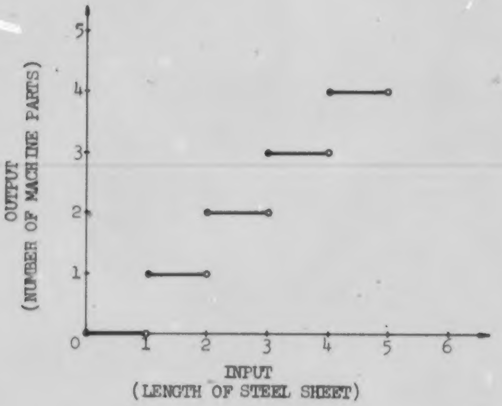


Figure 16

The tiny circle at the right of each horizontal segment shows us that this endpoint does not belong to the graph.

Exercises 2-7a

(Class Discussion)

1. We know, from Example 1, that the correspondence between number of hours worked and salary earned is a new kind of function called a step function.

   (a) When is a function called a step function?

   (b) In Example 2, is the correspondence between length of sheet steel and number of machine parts a step function?

   (c) In Figure 15, why are different scales used on the horizontal and vertical axes?

There are functions whose graphs are obtained by repeatedly sliding an initial portion of the graph to the right always by the same amount. Such a function is called a periodic function. The number of units by which we must slide the first portion to get the second and slide the second to get the third, and so forth, is called the period of the function. The function whose graph is drawn in Figure 17 is a periodic function whose period is 1.

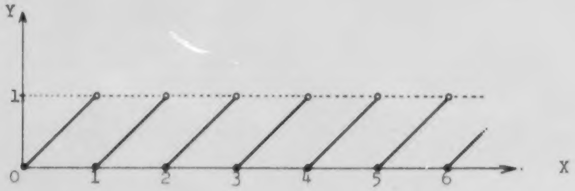


Figure 17

Note that over the interval from 0 to 1 the graph of this function is the same as that of the identity function. The rest of the graph can be thought of as being obtained by successively shifting this portion, between 0 and 1, one unit to the right. If a function has a period of 1, and x is an input, the output corresponding to an input of x + 1 is the same as the output associated with an input of x.

Of course, there are periodic functions with periods other than 1. For example, Figure 18 shows the graph of a periodic function with a period of 2.

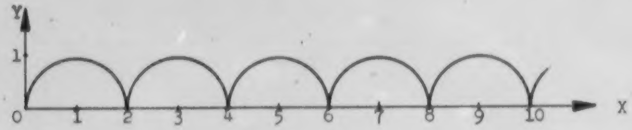


Figure 18

We can see that any one section of the graph must be shifted two units to the right to fit the next section of the graph.

The function whose graph is shown in Figure 19 is a periodic function whose period is 3. Any one section of the graph must be shifted three units to the right to fit the next section of the graph.

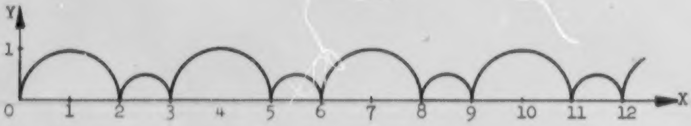


Figure 19

If a function has period 2, and x is an input, the output will be the same as the output for an input of $x + 2$. Likewise, if a function has period 3, and x is an input, the output will be the same as the output for an input of $x + 3$.

Exercises 2-7b

1. The Ace Car Rental Service rents a certain type of car for $7 per day with the agreement that use for any part of a day will carry a charge for the full day.

   (a) This statement involves a function. Below are 3 columns of correspondences belonging to this function. Study each column separately.

$\frac{1}{4} \longrightarrow 7$

$\frac{1}{2} \longrightarrow 7$

$1 \longrightarrow 7$

$1\frac{1}{4} \longrightarrow 14$

$1\frac{1}{2} \longrightarrow 14$

$1\frac{3}{4} \longrightarrow 14$

$2\frac{1}{24} \longrightarrow 21$

$2\frac{3}{24} \longrightarrow 21$

$3 \longrightarrow 21$

How do we know that this function is a step function?

(b) Draw the graph of this step function.

2. An electrical repair service charges $10 for the first hour and $6 per hour, or fraction of an hour, after the first hour.

   (a) Write 5 inputs and their corresponding outputs for this function.

   (b) Draw the graph of this function.

3. The cost of sending a letter by air mail in the United States is 10 cents per ounce or fractional part of an ounce. Draw a graph of this function.

4. Draw a graph of the function

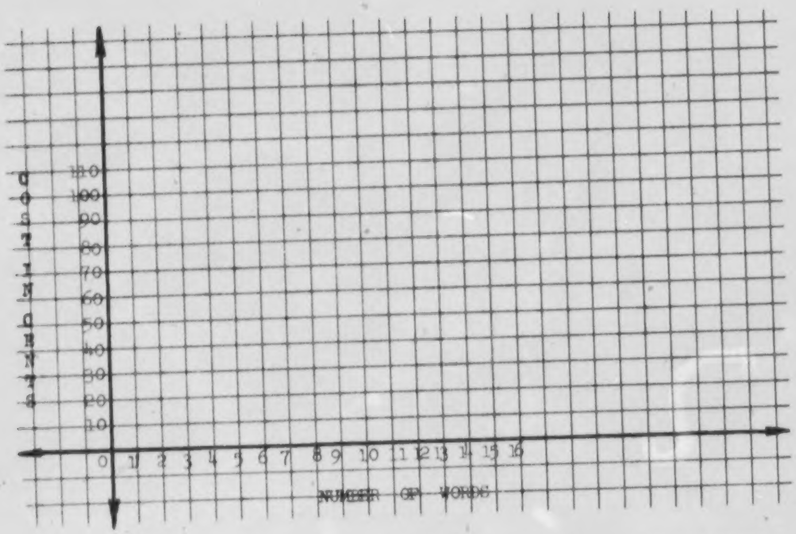
$$f : n \rightarrow \begin{cases} 1 & \text{if } 0 < n \leq 3, \\ 2 & \text{if } n > 3. \end{cases}$$

5. Consider the function described by the following statement:

> The cost of sending a telegram between two given points is 50 cents for the first 10 words (or less) and 5 cents for each word over 10.

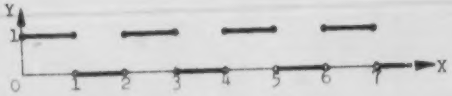
   (a) Represent this function using arrow notation.

   (b) Draw the graph of the function on a coordinate system like the one on the following page.

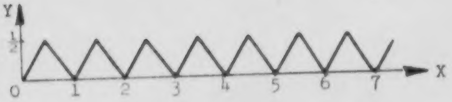


6. What are the periods of the functions whose graphs are shown below?

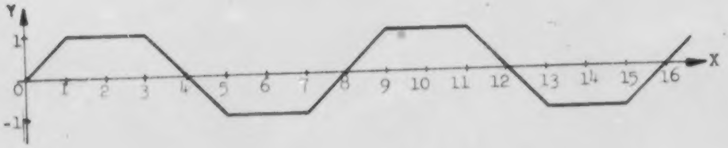
(a)



(b)



(c)



7. (a) The function whose graph is shown in 6(a) has two *different* outputs. What are the outputs for this function?

(b) The function whose graph is shown in 6(b) has the output $\frac{1}{2}$ corresponding to an indefinite number of different inputs. If x represents the smallest input whose output is $\frac{1}{2}$, then what counting number must be *repeatedly added to* x to find all other inputs which have the image $\frac{1}{2}$?

(c) Look at the graph shown in 6(c), and decide what output will correspond to *any* input between, and including, 21 and 23.

2-8. Summary

Section 2-1.

An introduction to functions was given in this section by way of a discussion on automobile travel using the familiar words, distance, speed, and time. For a given speed, we found that distance is a function of time; for example,

if the car travels 50 miles per hour, the distance, $d = 50t$;
if the car travels 45 miles per hour, the distance, $d = 45t$.

The average speed (in miles per hour) is the total distance traveled divided by the number of hours spent in travel.

Section 2-2.

If a heavy stone is dropped from a cliff or a building, it travels farther during the second one-second interval than during the first one, farther during the third one-second interval than during the second one, etc. In this way, a new rule, modeled on an actual physical problem, was introduced. Again, we have the distance as a function of time. The distance which a falling body, or falling object, drops can be approximated by

$$d = 16t^2.$$

A new symbol, ≈, meaning "approximately equal to", was introduced, and the above rule, or formula, is sometimes written

$$d \approx 16t^2.$$

Section 2-3.

A *function* is a special kind of correspondence such that corresponding to each input, there is exactly *one* output.

It is sometimes helpful to imagine a function as a kind of *machine*.

Arrow notation is a convenient way of representing a function in brief form.

The arrow notation *$f : n \to 2n$* can be read as "the function f which associates with each input n, the output 2n".

If a correspondence is a function, the *reverse* correspondence *may or may not* be a function.

Section 2-4.

The text has introduced you to *5* ways, other than arrow notation, for representing a function:

(1) a statement,

(2) a table,

(3) a diagram,

(4) a formula,

(5) a graph.

"To *draw* the *graph* of a function" means to find a collection of points in the coordinate plane that will accurately *picture* the special kind of association or correspondence which we call a *function*.

When drawing the graph of a function, *any* unit of measure may be selected for the X-axis, and *any* unit of measure for the Y-axis.

Section 2-5.

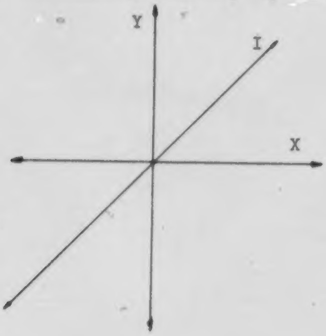
The *graph* of a *function* very often shows the correspondence more clearly than a table.

Certain *properties* of a *function* can be easily discovered by examining its graph.

Section 2-6.

A simple, but important, function is the identity function, I : x → x.

The graph of the identity function, I, is a line.



Section 2-7.

A step function is a function whose graph is made up entirely of a number of horizontal pieces.

A periodic function is a function whose graph can be drawn by finding that part of the graph which constantly repeats itself, and then sliding this part to the right always by the same amount.

Chapter 2

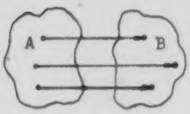
FUNCTIONS

This is the students' first formal introduction to the important mathematical concept of *function*. Since physical situations often lead to functions and graphs of functions the chapter also contains some introductory remarks about mathematical models.

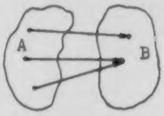
Although the *function concept* is one of the most significant ideas in the mathematical world, the textual treatment is merely a *gentle* introduction to the subject. *Please* treat it this way. That is, do *not* expect your students to master the concept *at this time*. The student will meet and work with the function concept through the rest of the sequence.

A *function* is a correspondence between two sets of things, usually numbers, such that to every member of the first set A there corresponds exactly one member of the second set B.

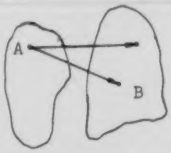
This statement can be illustrated as follows:



A function



A function



Not a function

Using the very descriptive words, *input* and *output*, we can say that for every *input* from set A there corresponds exactly one *output* in set B. The output is sometimes called the *image* of the member of set A.

We have initially chosen to emphasize the "arrow notation" to represent a function since it seems to convey to the student the basic feeling we would like him to gain about this concept. Refinements in the notation will occur as they are needed to handle more sophisticated situations. Do *not* introduce any additional notation at this time.

We usually write a formula which tells us the member of set B that corresponds to the member of the first set A. For example, if A is the set of all integers, and x is a member of set A, then

$$f : x \to x^2$$

is the function that takes any integer into its square. B is the set of all perfect squares 0, 1, 4, 9, 16, ...

Sometimes instead of a formula to denote the image of a member of set A we use a description of this image in words. For example, we may associate with each circle in the plane the single point which is its center and write

$$f : \text{circle} \to \text{center of circle}$$

to explain the function which we have in mind.

A brief version of the well-known story about Galileo is told not only to introduce an interesting function, but also to give the flavor of what discovery meant to Galileo, Newton and others. People have sought since ancient days to find a model of reality, to discover mathematical rules for such physical phenomenon as motion. Discovery meant finding a model, an ideal, or a mathematical description, which would approximate as nearly as possible a natural motion.[1]

In Galileo's experiment, we think of falling objects as points and we regard the earth as a plane. Our model for falling bodies is quite accurate for nearly spherical, dense objects dropped from a few hundred feet off the ground.

SUGGESTED TIME SCHEDULE:

Section	2-1	2-2	2-3	2-4	2-5	2-6	2-7
Days	1	1	1	2	2	1	2

---

[1] Jourdin, "The Nature of Mathematics". James Newman, ed., The World of Mathematics, ibid., pp. 44-5. Another reference which might be of interest to the teacher is Rudolph Carnap's Philosophical Foundations of Physics (N. Y. Basic Books, 1966).

2-1. Travel by Car

The student's introduction to functions is woven into a discussion of automobile travel. This illustration is chosen for two reasons:

(1) It is linear (a term not used with the students), and simple.

(2) This is a topic about which the students have some intuitive feeling, since this is usually a part of their previous experience.

Exercises 2-1a (Class Discussion)

1.

input t	output $d = 50t$
0	0
1	50
2	100
3	150
4	200
5	250

$d = 50t$

If $t = 0$, then $d = 50(0) = 0$

If $t = 1$, then $d = 50(1) = 50$

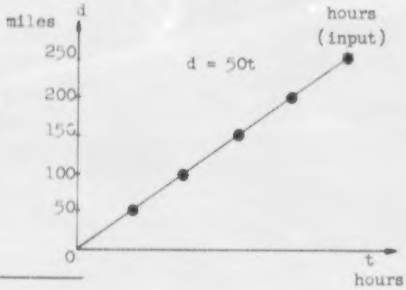
If $t = 2$, then $d = 50(2) = 100$

If $t = 3$, then $d = 50(3) = 150$

If $t = 4$, then $d = 50(4) = 200$

If $t = 5$, then $d = 50(5) = 250$

The students is not asked to plot his points on the coordinate axis, since this is not the emphasis in this section. The teacher might wish to return to this graph when the students are studying Section 2-4.



Exercises 2-1b (Class Discussion)

1. While the students have not had a "formula", as such, for computing average speed, this is a familiar area outside the classroom, and they can intuitively handle this important idea of average speed.

time	m/p/h	Total miles	Average speed
t = 1	65	65	65
2	50	115	$57\frac{1}{2}$
3	20	135	45
4	60	195	$48\frac{3}{4}$
5	55	250	50

Some of the values in the table above were given, but let us review how they were obtained and add the new ones:

During the first hour he drove 65 miles, averaging, of course, 65 miles per hour.

During the second hour, he drove an additional 50 miles, for a total of 115 miles. His average speed for the 2 hours would be $\frac{115}{2} = 57\frac{1}{2}$ miles per hour.

During the third hour, he drove only 20 miles, for a total of 135 miles. His average speed for the 3 hours would be $\frac{135}{3} = 45$ miles per hour.

During the fourth hour, he drove another 60 miles, for a total of 195 miles. His average speed for the 4 hours would now increase to $\frac{195}{4} = 48\frac{3}{4}$ miles per hour.

During the fifth hour, he drove an additional 55 miles, raising his average speed a little, to $\frac{250}{5} = 50$ miles per hour.

2. (a) No, Bob will always be eight years older than John. This can be seen from the graph.

   (b) Yes, when John was 8 years old and Bob was 16, Bob was twice the age of John, in 1958.

(c) Yes, when John was 4 years old and Bob was 12, Bob's age was three times that of John's, in 1954.

(d) Yes, when John was 2 years old and Bob was 10, Bob's age was five times that of John's, in 1952.

(e) Yes, when John was a year old and Bob was 9, Bob's age was nine times that of John's, in 1951.

2-2. Falling Objects

The story of Galileo is told to help introduce a function which is perhaps a little different from the ones the students have intuitively dealt with before. In this story the idea of functional dependence (function of) is encountered before the actual function is ascertained. Incidentally, the function concept is considered to have originated with Galileo.

After graduating from "functional dependence" to the actual function, we immediately turn to the application of measuring the height of a waterfall with a stop watch.

Although "domain" (the set of inputs) and "range" (the set of outputs) are not mentioned, we stress the idea of input and output where the input value determines the output value. The last part of this section focuses upon the variables which have been ignored in the earlier discussion, and brings to light the model involved in a mathematical description of a real-life situation.

Exercises 2-2.

1. t : number of seconds

If t =	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
then d =	0	4	16	36	64	100	144	196

$$d = 16t^2$$

If $t = 0.0$, then $d = 16(0)^2 = 16(0) = 0$

If $t = 0.5$, then $d = 16(.5)^2 = 16(.25) = 4$

If $t = 1.0$, then $\underline{d = 16(1)^2 = 16(1) = 16}$

If $t = 1.5$, then $d = 16(1.5)^2 = 16(2.25) = 36$

If $t = 2.0$, then $\underline{d = 16(2)^2 = 16(4) = 64}$

If $t = 2.5$, then $d = 16(2.5)^2 = 16(6.25) = 100$

If $t = 3.0$, then $\underline{d = 16(3)^2 = 16(9) = 144}$

If $t = 3.5$, then $d = 16(3.5)^2 = 16(12.25) = 196$

2. Computing differences from the table of problem 1 above, we have

t =	0	.5	1	1.5	2	2.5	3	3.5
d =	0	4	16	36	64	100	144	196
difference in d		4	12	20	28	36	44	52

Therefore:

during the $\frac{1}{2}$ - second intervals,	the rock fell
t = 0 to t = .5	4 ft.
t = .5 to t = 1	12 ft.
t = 1 to t = 1.5	20 ft.
t = 1.5 to t = 2.0	28 ft.
t = 2 to t = 2.5	36 ft.
t = 2.5 to t = 3.0	44 ft.
t = 3 to t = 3.5	52 ft.

3. If we take differences again we have (repeating the tables of problems 1 and 2, and extending them),

t =	0	.5	1	1.5	2	2.5	3	3.5	4	4.5	5
d =	0	4	16	36	64	100	144	196			
differences of distance	4	12	20	28	36	44	52	60	68	76	
differences of differences	8	8	8	8	8	8	8	8	8		

That is, if we observe the difference between the numbers representing the number of feet that the rock fell in consecutive half-second intervals, we will note a constant difference of 8. From this 8, we can extend the table by adding; i.e.,

The rock fell $52 + 8 = 60$ ft. between $t = 3.5$ and $t = 4$;
it fell $60 + 8 = 68$ ft. between $t = 4$ and $t = 4.5$;
it fell $68 + 8 = 76$ ft. between $t = 4.5$ and $t = 5$.

2-3. Some Examples of Functions

This section really has a four-fold objective. It introduces the students to:

(1) a few varied functions,

(2) the machine interpretation of a function,

(3) the arrow notation method of representing a function,

(4) the reversing of the direction of correspondences.

The class discussion exercises are designed to review certain fundamental ideas from geometry as well as to test whether students can recognize the statement of a functional relationship.

Assign Exercises 2-3b. When discussing these exercises with the students, emphasize ideas rather than language.

Exercises 2-3a (Class Discussion)

1. (a) This statement does *not* describe a function. Given a point in space, there are an *un*limited number of lines containing it.

   (b) This statement describes a function. Through any two different points in space there is exactly *one* line. $f : (A,B) \rightarrow \overleftrightarrow{AB}$.

   (c) This statement does *not* describe a function. Given a point in space, there are an *un*limited number of planes containing it.

   (d) This statement does *not* describe a function. Given two different points in space, there are an *un*limited number of planes containing them.

   (e) This statement describes a function. Through any three points, not all on the same line, there is exactly *one* plane.
   $f : (A,B,C) \rightarrow$ plane ABC.

Exercises 2-3b

1. (a) Associate with each member of your mathematics class his or her age in years.

   This statement describes a function.
   f : member → age of member

   Associate with each age, in years, a member of your mathematics class.

   This statement does *not* describe a function.

   (b) Associate with each state of the United States the name of its *present* governor.

   This statement describes a function.
   f : state → name of *present* governor

   Associate with the name of each *present* governor in the United States the state which he serves.

   This statement describes a function.
   g : name of *present* governor → state

   (c) Associate with each height, in inches, a citizen of the United States.

   This statement does *not* describe a function.

Associate with each citizen of the United States his or her height in inches.

This statement describes a function.

f : citizen $\rightarrow$ height of citizen

(d) Associate with a first name for a person, a last name.

This statement does *not* describe a function.

Associate with a last name for a person, a first name.

This statement does *not* describe a function.

2-4. *Ways of Representing Functions*

The objective of this section is to introduce the students to significant (1) *ideas*, (2) *symbolism*, and (3) *vocabulary*, relevant to the *function* concept.

It is *absolutely essential* that the students *understand* the *content* of this section. There are *two* major reasons for this statement.

(1) It *prepares* the students to cope with the remaining sections of this chapter.

(2) It *also* prepares them for *much* of their *future work in mathematics*.

Although this section is not explicitly called a class discussion exercise, it should be treated as one, but *only* after the students have been instructed to *carefully read* the *entire* section.

The text introduces the students to *5* methods, other than arrow notation, for representing a function. Each of these ways should be discussed and *in the order* of presentation: (a), ..., (e). Give special attention to part (e), the *graph*. There are many subtle, and *significant*, ideas involved in this exposition.

Concerning *Exercises 2-4*, questions 8 and 9 will probably be most difficult for the students (5 and 7 to a lesser degree). *All* questions should be assigned, but especially *7* as it introduces the *constant function*, a function which the student will frequently meet in subsequent sections.

Exercises 2-4

1. (a) $n : c \rightarrow 3c$, c represents a counting number.

   (b) No. There are an unlimited number of correspondences.

2. (a) 3 (b) 5 (c) 13

   (d) $f : c \rightarrow c + 3$, c represents a counting number.

3. (a) 9 (b) 7

   (c) No. If w represents a whole number, then $2w$ represents an even number. Therefore, $2w + 1$ will represent an odd number.

4. (a) $d = 40t$

   (b) $g : t \rightarrow 40t$, $t > 0$

   (c) 140,

   $$40 \cdot 3\frac{1}{2} =$$
   $$40 \cdot (3 + \frac{1}{2}) =$$
   $$(40 \cdot 3) + (40 \cdot \frac{1}{2}) =$$
   $$120 + 20 =$$
   $$140$$

   (d) $3\frac{3}{4}$, If $40t = 150$, then $t = \frac{150}{40}$.

   $$\frac{150}{40} = \frac{10 \cdot 15}{10 \cdot 4} = \frac{10}{10} \cdot \frac{15}{4} = \frac{15}{4} = 3\frac{3}{4}$$

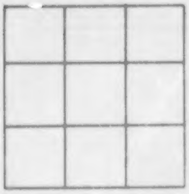
5. (a) 9, 36. Since $4 \cdot 9 = 36$, the second output is four times the first.

   (b) 16, 64. Since $4 \cdot 16 = 64$, the second output is four times the first.

   (c) "It seems to hold" is an acceptable answer.

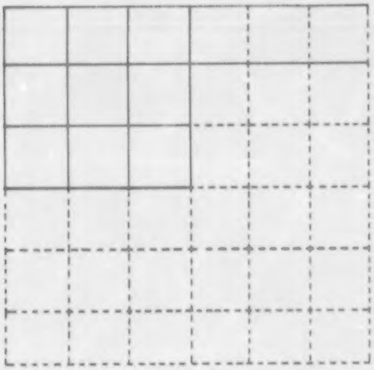
   The teacher should know that it always holds.

3 units



$3 \to 9$

6 units



$6 \to 36$

$s \to s^2$

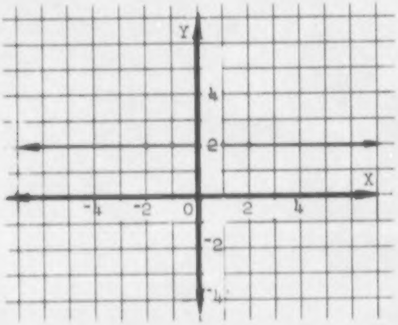
$2s \to (2s)^2$, but $(2s)^2 = (2s) \cdot (2s) = (2 \cdot 2) \cdot (s \cdot s) = 4s^2$

6. (a) 3 (b) 6 (c) 3

(d) Yes. The graph is rising from left to right.

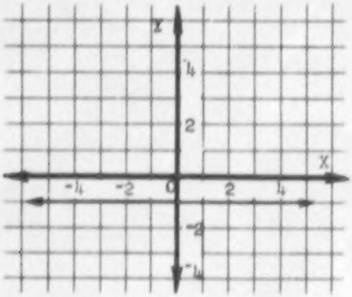
7. (a)

Input: x	$^-2$	$^-1$	0	1	2
Output: 2	2	2	2	2	2



(b)

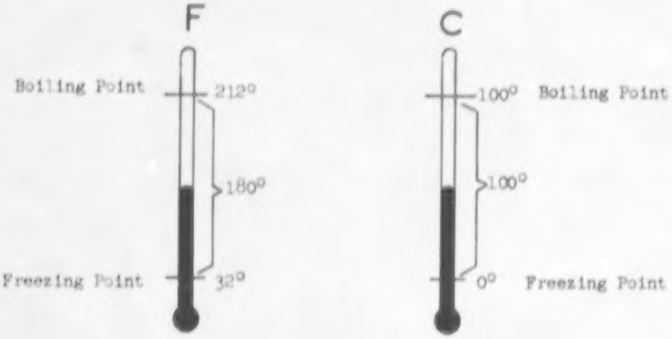
Input: x	$^{-}2$	$^{-}1$	0	1	2
Output: $^{-}1$	$^{-}1$	$^{-}1$	$^{-}1$	$^{-}1$	$^{-}1$



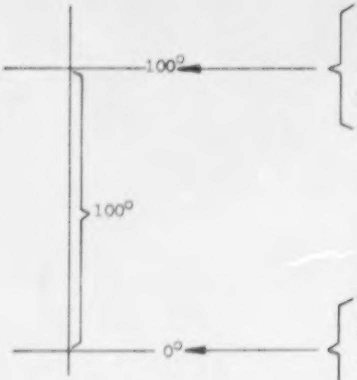
The graph of each function is a line, and both lines are horizontal (parallel to the X-axis).

The function, k, described by $k : x \rightarrow 2$, is such that regardless of the input chosen, the corresponding output is constantly 2 (likewise for the function ℓ).

8.


F
C
Boiling Point
212°
180°
32°
Freezing Point
100°
Boiling Point
100°
0°
Freezing Point

Thermometers are available in both F and C scales. The relation is due to the markings in these scales.



This is the point on the Celsius (Centigrade) Scale where water is converted into steam (with additional heat) and steam is converted into water (with removal of heat).

This is the point where water is converted into ice (with removal of heat) and ice is converted into water (with additional heat).

(a) 32, freezing point of water

(b) 212, boiling point of water

(c)

Input: C	0	10	40	60	80	100
Output: $\frac{9}{5}C + 32$	32	50	104	140	176	212

If C = 0,
then $\frac{9}{5}C + 32 =$
$\frac{9}{5} \cdot 0 + 32 =$
$0 + 32 =$
32

If C = 10,
then $\frac{9}{5}C + 32 =$
$\frac{9}{5} \cdot 10 + 32 =$
$18 + 32 =$
50

If C = 40,
then $\frac{9}{5}C + 32 =$
$\frac{9}{5} \cdot 40 + 32 =$
$72 + 32 =$
104

If C = 60
then $\frac{9}{5}C + 32 =$
$\frac{9}{5} \cdot 60 + 32 =$
$108 + 32 =$
140

If $C = 80$,

then $\frac{9}{5}C + 32 =$

$\frac{9}{5} \cdot 80 + 32 =$

$144 + 32 =$

176

If $C = 100$,

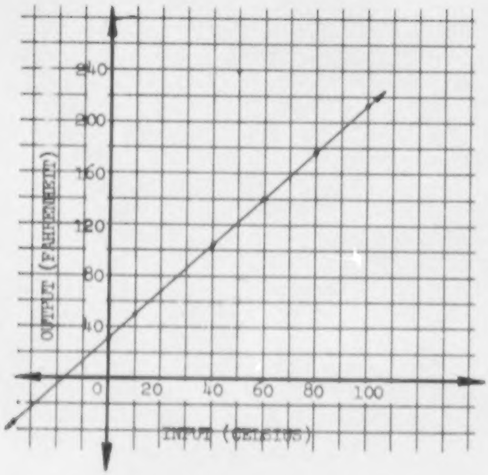
then $\frac{9}{5}C + 32 =$

$\frac{9}{5} \cdot 100 + 32 =$

$180 + 32 =$

212

(d)



Concerning this graph, some students will guess that the points which they have plotted are collinear, others will guess they are not collinear. In terms of proof neither group can justify their guess. It is hoped that the majority of students will agree that, at this time, they really don't know. At a future time, the collinearity of these points will be proved.

9. (a)

Input: V	1	2	4	5	10	20	25	50
Output: $\frac{50}{V}$	50	25	$\frac{25}{2}$	10	5	$\frac{5}{2}$	2	1

If $V = 1$,

then $\frac{50}{V} =$

$\frac{50}{1} =$

50

If $V = 2$,

then $\frac{50}{V} =$

$\frac{50}{2} =$

25

If $V = 4$,

then $\frac{50}{V} =$

$\frac{50}{4} =$

$\frac{2 \cdot 25}{2 \cdot 2} =$

$\frac{2}{2} \cdot \frac{25}{2} =$

$\frac{25}{2}$

If $V = 5$,

then $\frac{50}{V} =$

$\frac{50}{5} =$

10

If $V = 10$,

then $\frac{50}{V} =$

$\frac{50}{10} =$

5

If $V = 20$,

then $\frac{50}{V} =$

$\frac{50}{20} =$

$\frac{10 \cdot 5}{10 \cdot 2} =$

$\frac{10}{10} \cdot \frac{5}{2} =$

$\frac{5}{2}$

If $V = 25$,

then $\frac{50}{V} =$

$\frac{50}{25} =$

2

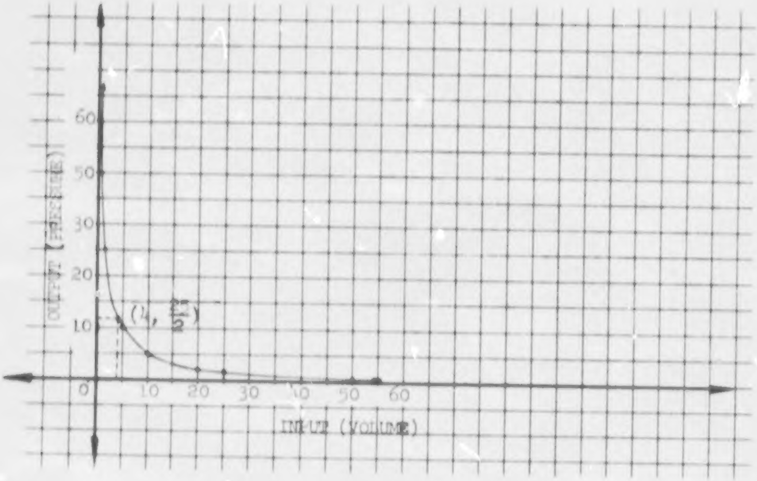
If $V = 50$,

then $\frac{50}{V} =$

$\frac{50}{50} =$

1

(b)



Concerning this graph, *all* students will *know* that the points which they have plotted are *not* collinear. But, by drawing a *smooth curve* through the plotted points, a fairly accurate sketch of the true graph of the function can be drawn.

(c) No. The graph of function g is *falling* from left to right. Therefore, function g is a *decreasing function*.

10. (a), (b), (c).

2-5. *Discovering the Usefulness of Graphs*

In the preceding section (2-4.) the students learned that "a *graph* is one of the most useful ways of representing a function." The purpose of this section is to further develop the *meaning* implicit in this statement.

A class discussion exercise has been inserted into this section at a *critical* time, and it is *imperative* that it be *carefully exploited*. It is designed to increase the students' awareness of the kinds of information which are readily available from the graph of a function if it is examined in an appropriate way.

This section will be helpful to students in developing some understanding of the basic reasons for obtaining functions and their graphs. The analysis of functions and their graphs produces information of a "global" sort not usually apparent from the initial observation of the functional relationship.

Some of the main ideas in this section are:

(1) The graph of the function very often displays the functional relationship more clearly than a table. (In this section Tom's growth pattern shows up more clearly on the graph.)

(2) The identification of some properties of a function can be made more readily from the graph of the function.

(3) Properties of the function are discernible from the graph and can be used in a predictive fashion to indicate what the pattern might be like beyond the tabular values.

Concerning Exercises 2-5b, all questions should be assigned. None are especially difficult.

A second set of questions, Exercises 2-5c, immediately follows the first set. No new knowledge is needed by the students in order to answer these questions. But, in question 4, the students are directed to make a slight, but significant, extension of their existing conceptual framework. Their intuition should accomplish this. The questions are intended, primarily, to reinforce the students' understanding of their newly acquired concepts. The entire set should be assigned, but only after the previous set has been assigned and discussed.

Exercises 2-5a (Class Discussion)

1. (a) Rising. An increase in the input produces an increase in the output.

   (b) An increasing function.

   (c) As Tom grows older, from birth to 18, he also grows taller.

2. (a) In terms of rising or falling, nothing is happening to the graph. A change in the input produces no change in the outputs.

(b) As Tom grows older, from 18 to 21, he *does not* grow taller.

(c) For the inputs (ages), from 18 to 21, the function, h, behaves like a *constant* function. (Refer the students to question 7, Exercises 2-4.)

3. (a) Yes.

(b) There is a 1 year period, between the ages of 0 and 21, in which Tom *grew more* than in any other 1 year period within this interval.

(c) Yes.

(d) Coordinates of the left endpoint: (17,64)
Coordinates of the right endpoint: (18,71)

(e) Between his 17th and 18th birthdays.

(f) 71 - 64, 7 inches

(g) Yes. The *steepness* of the segment joining the points whose coordinates are $(16, 58\frac{1}{4})$ and (17,64) *indicates* that Tom *grew more* between his 16th and 17th birthdays than in any other 1 year period, *excluding* the period between his 17th and 18th birthdays.

(h) $64 - 58\frac{1}{4}$; $5\frac{3}{4}$ inches, $64 - 58\frac{1}{4} = 63\frac{4}{4} - 58\frac{1}{4} = 5\frac{3}{4}$

Exercises 2-5b

1. Between birth and his 1st birthday, 0 and 1,
$25\frac{1}{4} - 21 = 4\frac{1}{4}$ ($4\frac{1}{4}$ inches)

2. Between his 18th and 19th birthdays, 18 and 19,
71 - 71 = 0 (0 inches)

Between his 19th and 20th birthdays, 19 and 20,
71 - 71 = 0 (0 inches)

Between his 20th and 21st birthdays, 20 and 21,
71 - 71 = 0 (0 inches)

3. (a) His 9th birthday, his 17th birthday

(b) $\frac{71}{21}$

$$\frac{71}{21} = \frac{63}{21} + \frac{8}{21} = 3 + \frac{8}{21} = 3\,\frac{8}{21}$$

3.38

4. (a)

Age	1	2	3	4	5	6	7	8	9	10
Growth since preceding birthday	$4\frac{1}{4}$	$3\frac{1}{2}$	$2\frac{1}{2}$	$2\frac{1}{4}$	2	$1\frac{3}{4}$	2	$1\frac{3}{4}$	3	$2\frac{1}{2}$

Age	11	12	13	14	15	16	17	18	19	20	21
Growth since preceding birthday	$2\frac{1}{4}$	$2\frac{1}{4}$	$1\frac{3}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	$2\frac{1}{4}$	$5\frac{3}{4}$	7	0	0	0

(b) 0 is an acceptable input for the function h because the function associates the age, in this case 0, with the height at birth for the age.

0 → 21

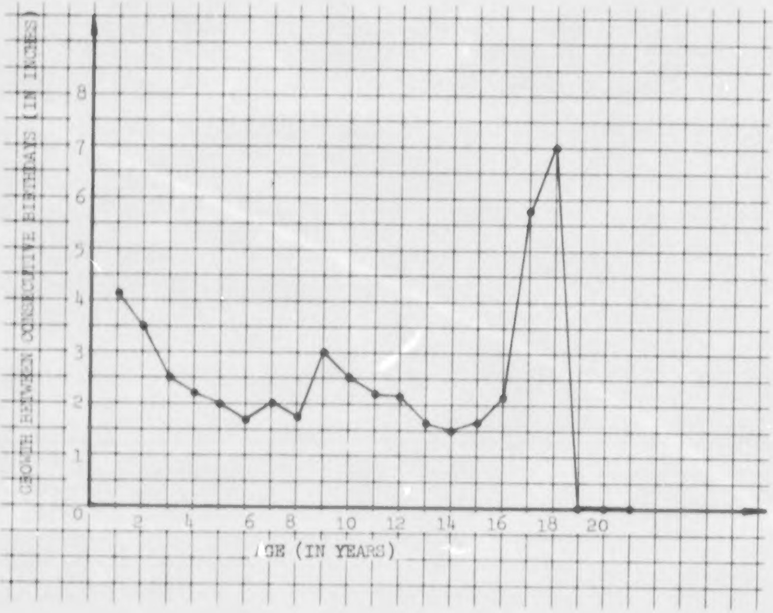
0 is not an acceptable input for the function g because the function associates age, in this case 0, with the growth between preceding birthday and birthday for the age. There is no preceding birthday for age 0.

(c) Yes. 21 is an acceptable input for our function g, since g associates age with the growth between preceding birthday and birthday for the age.

21 → 0

The output is 0, because the growth between 20 and 21 is 0.

5. (a)



(b) No.

(c) We know that as Tom grows older, specifically from birth to age 6, he also grows taller. The interpretation to be made from examining the graphs of the functions is that while Tom is growing taller, within this age interval, he is doing so at a decreasing rate of growth.

(d) We know that as Tom grows older, specifically from 13 to 18, he also grows taller. The interpretation to be made from examining the graphs of the functions is that while Tom is growing taller, within this age interval, he is doing so at an increasing rate of growth.

6. Average growth per year $= \dfrac{\text{Total growth}}{\text{Number of years}}$

$$= \frac{71 - 21}{18}$$

$$\frac{71 - 21}{18} = \frac{50}{18} = \frac{2 \cdot 25}{2 \cdot 9} = \frac{2}{2} \cdot \frac{25}{9} = \frac{25}{9} = 2\,\frac{7}{9}$$

$2\,\frac{7}{9}$ inches

Exercises 2-5c

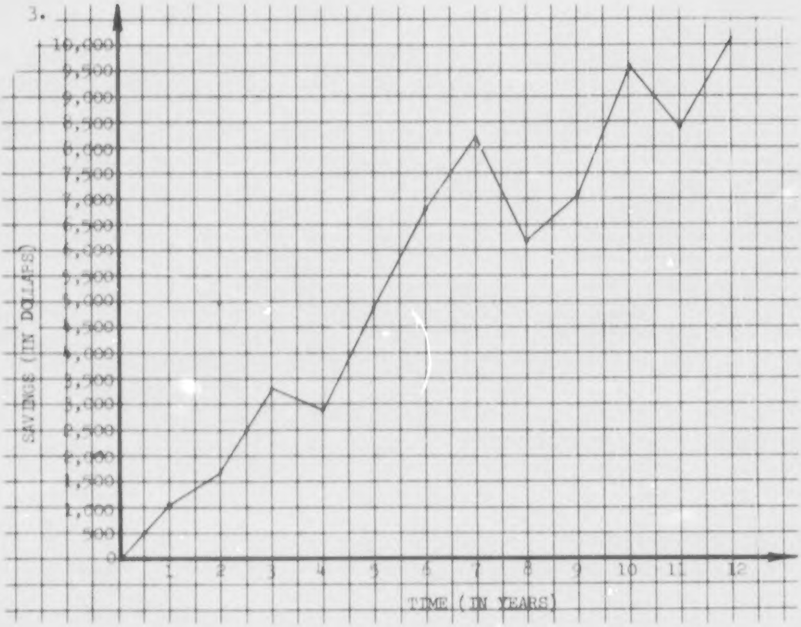
1. (a) By previous agreement, the notation (e.g., 1 → 1020) implies this.

   (b) Outputs or images

2. (a) Quadrant I (the 1st quadrant)

   (b) Join consecutive points, belonging to the graph of the function, by segments.

3.



4. A *decrease* in savings

5. The 10th year. The segment, joining the points whose coordinates are (9, 7028) and (10, 9560) is *steeper* than any other segment which *rises* from left to right.

6. The 8th year. The segment, joining the points whose coordinates are (7, 8120) and (8, 6215), is *steeper* than any other segment which *falls* from left to right.

7. 3 years. If a segment *rises* from left to right, then this indicates an *increase* in savings. The maximum number of *consecutive segments* which do this is 3. There are *2* 3-year periods during which there were *only increases* in savings, the 1st 3 years and from the 5th year through the 7th year.

8. The segment, joining the points whose coordinates are (3, 3312) and (4, 2906) *falls* from left to right, and this indicates a *decrease* in savings (i.e., a withdrawal of savings).

9. The segment, joining the points whose coordinates are (9, 7028) and (10, 9560), *rises* from left to right, and this indicates an *increase* in savings. From question 5, we *also* know that this *increase* in savings is *maximum* over the 12-year interval.

10. (a) $10,164

    (b) $\frac{10164}{12} = 847$

    $847

2-6. The Identity Function

In this section the students are introduced to a *significant* function, viz., the *identity function*.

The exposition concerning the folding of a square-shaped item, such as a napkin, is excellent. It should lead to a *visual demonstration* that will help relate geometrical and functional concepts, and thus contribute to a better understanding of mathematics. The word "bisects" appears in the text. At this time, do *not* attempt a formal definition, but be prepared to discuss the idea informally.

Exercises 2-6 are not difficult, but they are important. Both questions should be assigned.

1. (a)

Input: x	$^-2$	$^-1$	0	1	2
Output: x	$^-2$	$^-1$	0	1	2

(b)

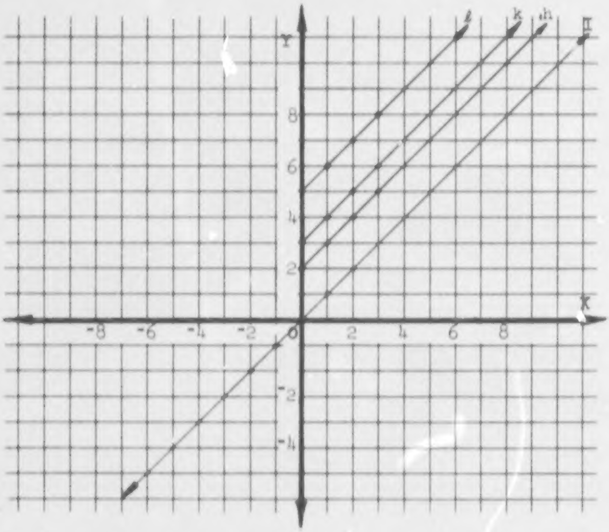
Input: x	0	1	2	3
Output: x + 2	2	3	4	5

(c)

Input: x	0	1	2	3
Output: x + 3	3	4	5	6

(d)

Input: x	0	1	2	3
Output: x + 5	5	6	7	8



2. (a) All 3 graphs are rays.

   (b) Each ray is parallel to the line.

   (c) Yes. The graph of each function is rising from left to right.

2-7. New Kinds of Functions

This section will acquaint the students with 2 new kinds of functions, the step function and the periodic function.

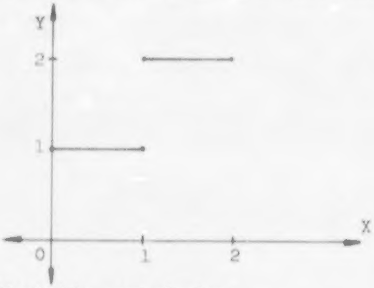
A class discussion exercise has been inserted into this section. It serves a dual role. Step functions are introduced first, and this exercise attempts to reinforce learning relevant to this kind of function. It also serves to separate the textual presentation of one function from that of the other.

The step function provides an early exposure to discontinuous functions, which play an important part in some areas of mathematics. Some topics for brief discussion and emphasis with students include:

(1) The reason we call it a "step" function. This is rather obvious, but students might like to talk about it a little.

(2) The fact that this is a function--i.e., for each input there is a single output.

(3) The fact that the graph is "made up . . . of horizontal pieces" but that the pieces are not segments, since each lacks an endpoint. If this were not so, then at each value of x where one horizontal piece ends and the next begins we would have one input with two outputs. For example, the graph indicated here is not the graph of a function, since for the input 1 it shows outputs of both 1 and 2.



(4) "Missing" endpoints are indicated by open circles.

Another special function is the periodic function. A physical application of a periodic function is to consider a red spot on the wheel of a car. What is the graph of the motion and position of this spot as the car moves forward? Certainly there are numerous examples of periodic functions and the discussion of this subject here is only an introduction.

Concerning Exercises 2-7b, it is again recommended that all questions be assigned.

Exercises 2-7a (Class Discussion)

1. (a) A function is named a step function if its graph is made up entirely of a number of horizontal pieces. This is the fundamental characteristic of this kind of a function.

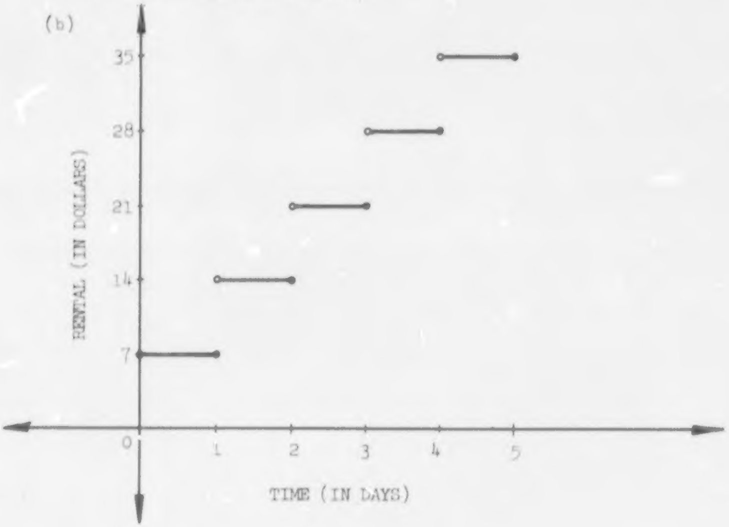
   (b) Yes.

   (c) For convenience in representing the data.

Exercises 2-7b

1. (a) The 3 columns of correspondences, belonging to this function, imply that the graph of the function is made up completely of a number of horizontal pieces.

   (b)



2. (a) input → output

Examples:

$\frac{1}{2} \rightarrow 10$

$\frac{3}{4} \rightarrow 10$

$1 \rightarrow 10$

$1\frac{1}{4} \rightarrow 16$

$1\frac{1}{3} \rightarrow 16$

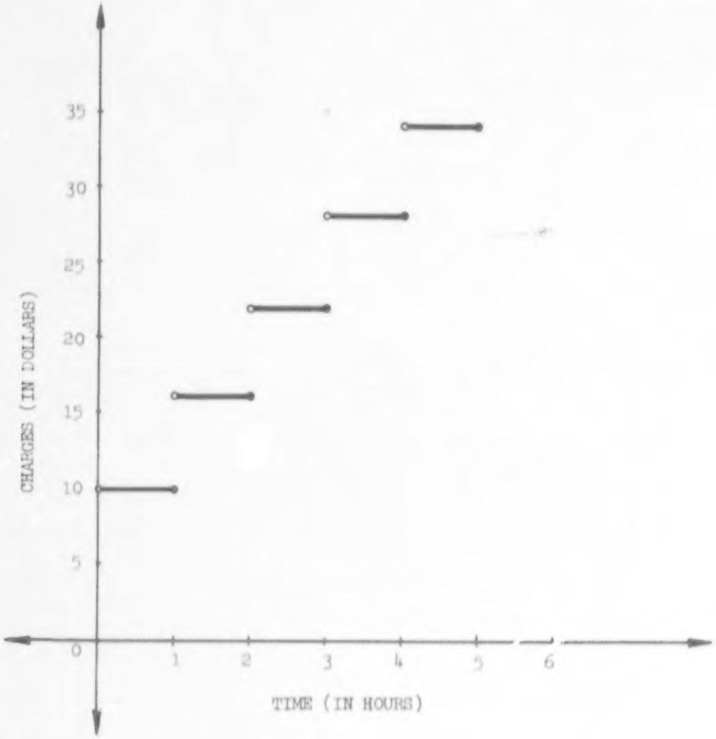
$2 \rightarrow 16$

$2\frac{1}{6} \rightarrow 22$

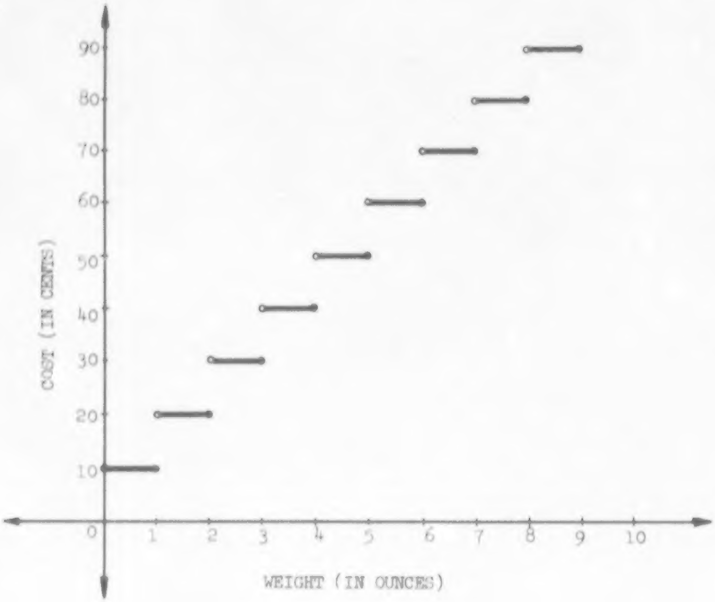
$2\frac{1}{2} \rightarrow 22$

$2\frac{3}{4} \rightarrow 22$

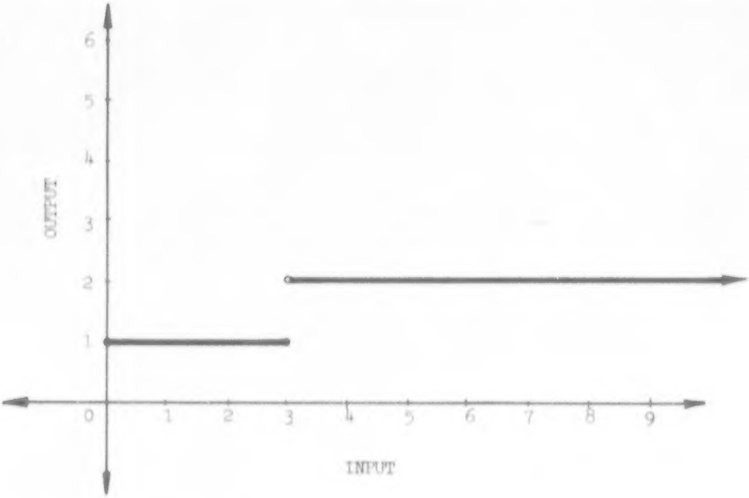
(b)



3.


COST (IN CENTS)
90
80
70
60
50
40
30
20
10
0
1
2
3
4
5
6
7
8
9
10
WEIGHT (IN OUNCES)

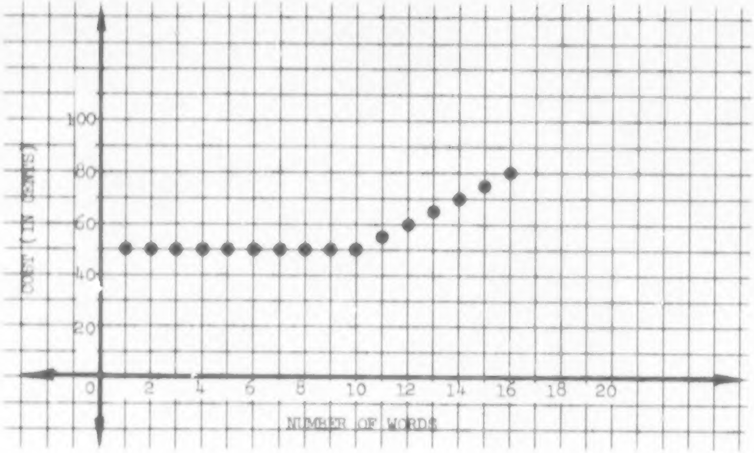
4.


OUTPUT
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
INPUT

5. (a)

$$f : n \to \begin{cases} 50 & \text{if } n \text{ belongs to } \{1, 2, 3, \ldots, 10\}. \\ 50 + 5(n - 10) & \text{if } n \text{ belongs to } \{11, 12, 13, \ldots\}. \end{cases}$$

(b)



6. (a) 2 (b) 1 (c) 8

7. (a) 0 and 1

(b) 1

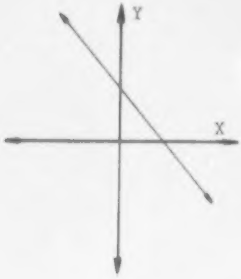
(c) $^-1$, $21 - \underline{8} = 13$ and $23 - \underline{8} = 15$

The inputs, 13 and 15, appear in drawing 5(c).

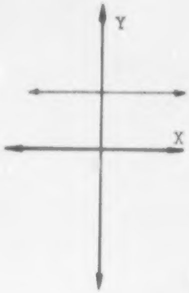
Suggested Test Items

Multiple Choice

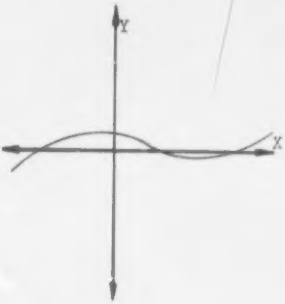
1. Identify the graph which does not represent a function.



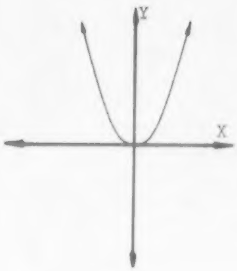
(A)



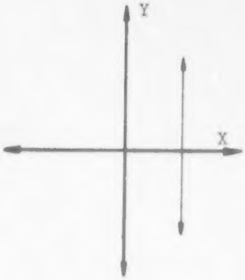
(B)



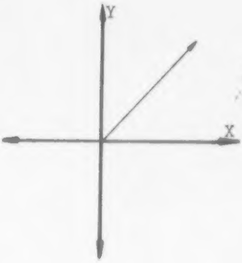
(C)



(D)

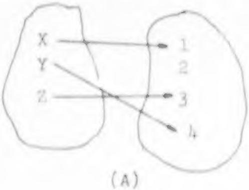


(E)

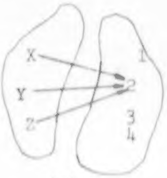


(F)

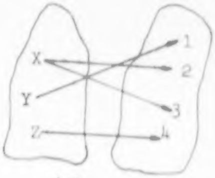
2. Identify the diagram which does not represent a function.



(A)

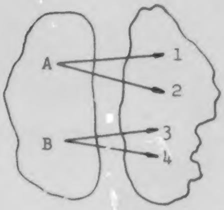


(B)

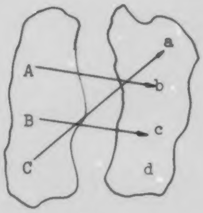


(C)

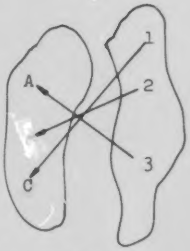
3. Identify the diagrams which represent functions.



(A)

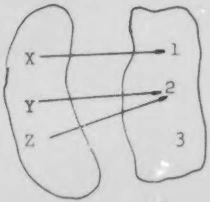


(B)

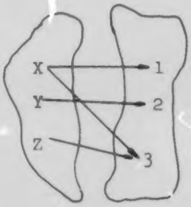


(C)

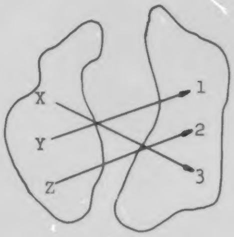
4. Each of the diagrams below shows a correspondence. Identify the diagram that represents a function and which will still do so if the correspondence is reversed.



(A)



(B)



(C)

Completion

1. A function is a special kind of ______________________.

2. A function whose output value is the same for each input value in the set of inputs is said to be a _________________ function.

3. The perimeter (p) of a square is four times the length of one side (s).

   (a) Express this function by a formula. ______________________

   (b) Express this function, using the arrow notation.

   ____

(c) Write a mathematical sentence which describes the set of acceptable inputs for this function. ______________

(d) An input of 9 determines an output of ________.

(e) An output of 2 results from an input of ________.

4. Two numbers are so related that one is five times the other.

(a) Does this statement describe a function? ________

(b) Name the input with a letter. ________

(c) Name the output. ________

(d) Write the arrow notation indicating this relationship.

____

(e) If the input is 7, what is the output? ________

5. Given the function, f, described by $f : x \rightarrow 3x + 2$, find the output corresponding to each input in the table below.

Input: x	0	1	2	3	4	5	6	7
Output: $3x + 2$								

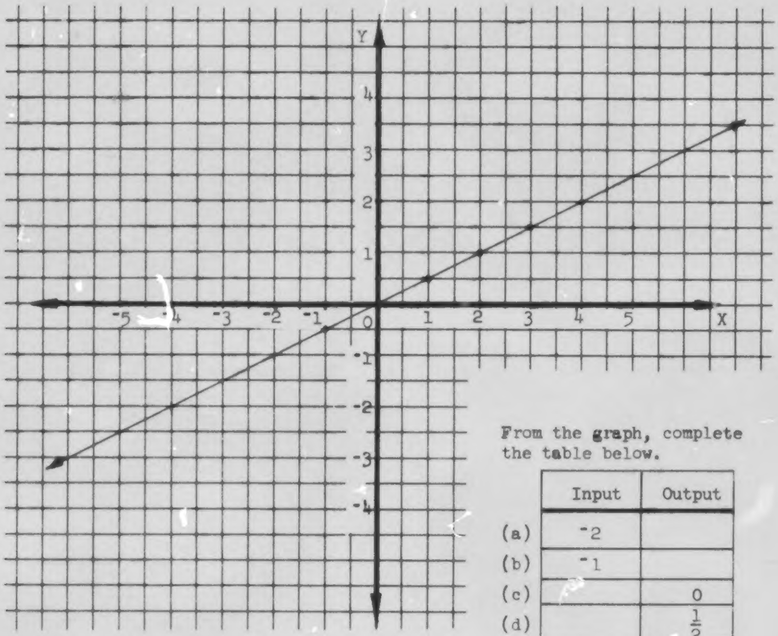
6. (a) The velocity (speed) of a falling body can be found by using the formula $v = 32t$. Express this function by using arrow notation, and name the function f. ______________

(b) In the formula $v = 32t$, which letter refers to the set of inputs of the function? ________

(c) The formula $v = 32t$ informs us that an input of 10 produces an output of ________, and an output of 256 must have been produced by an input of ________.

(d) Write a mathematical sentence indicating acceptable inputs for the formula $v = 32t$. ______________

7. The graph below represents a function.



From the graph, complete the table below.

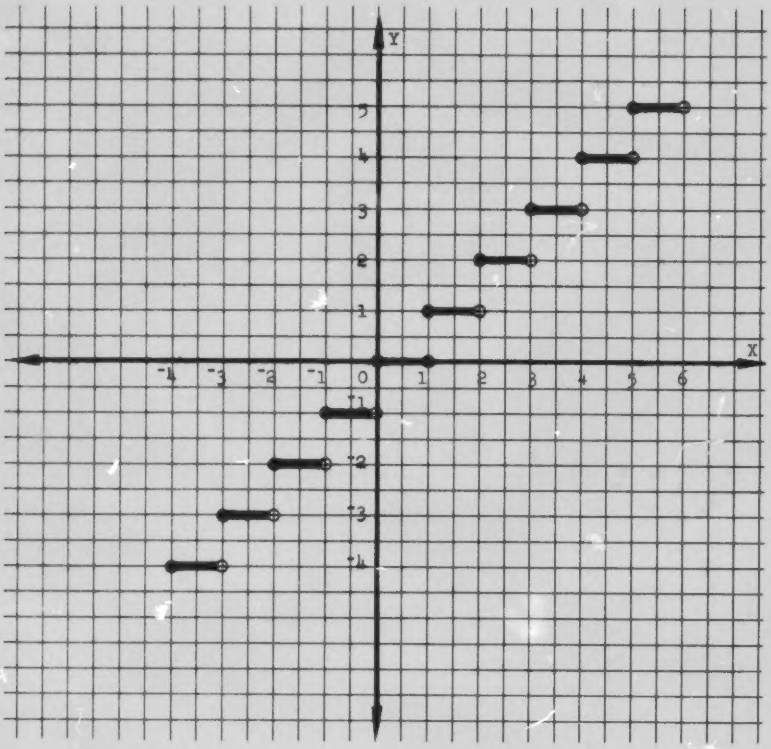
	Input	Output
(a)	$^-2$	
(b)	$^-1$	
(c)		0
(d)		$\frac{1}{2}$
(e)	2	

(f) What will be the output for an input of n? __________

(g) Describe this function using arrow notation.

____

8. The graph below represents a function.

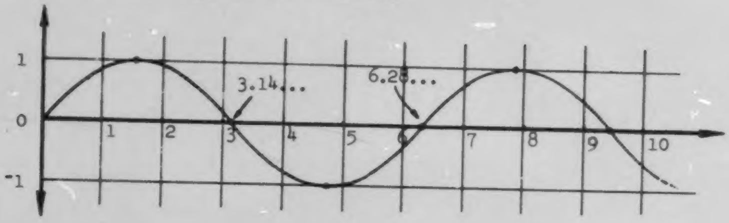


(a) What name is given to a function that has a graph as is pictured above? ____________________

(b) From the graph, fill in the following table.

Input	⁻3	⁻2.25	⁻1 $\frac{1}{2}$				4 $\frac{1}{4}$	5	5 $\frac{1}{2}$
Output				0	1	1			

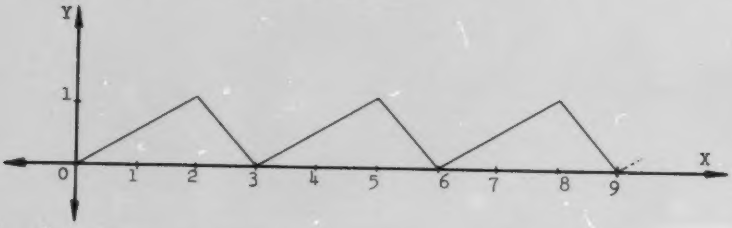
9. The graph below represents a function.



(a) This is the graph of a __________________ function.

(b) How far must you shift any one section of this graph to fit the next section of the graph? _________ units

10. The graph below represents a function.



(a) This is the graph of a __________________ function.

(b) How far must you shift any one section of this graph to fit the next section of the graph? _________ units

Problems

1. Each statement below specifies a correspondence. Some specify functions, some do not. For each statement that specifies a function, use arrow notation to show the input and the corresponding output.

Example: Associate with each article in a store its selling price. g : article → price of article

(a) Associate with a father his four children.

(b) Associate with each student in this mathematics class, who took the test, his or her letter grade on the last test.

(c) Associate with each legal driver in the state of California his or her drivers license number.

(d) Associate with each ordered pair (x,y) on a coordinate plane the corresponding point P.

(e) Associate with each non-negative integer (whole number) a positive integer (counting number) of equal value.

Reverse the correspondence in each of the statements above, and for each new statement that specifies a function use arrow notation to show the input and the corresponding output.

(a)

(b)

(c)

(d)

(e)

2. For the correspondences shown in each table below, answer the following questions.

(1) Is this correspondence a function?

(2) Is the reversed correspondence a function?

(a)

Input	-4	-2	0	2	4	6
Output	0	1	4	9	16	25

(b)

Input	-1	0	1	2	4	6
Output	-1	0	1	2	4	6

(c)

Input	18	8	2	0	2	8
Output	-4	-2	0	2	4	6

3. In the formula $h = \frac{2}{3} d^2$,

h is the distance above the ground (measured in feet) and
d is the approximate distance to the horizon (measured in miles).

Thus, if a person wishes to see a distance of d miles over level ground, he must be h feet above the ground.

(a) Write this relationship as a function, using the arrow notation.

(b) What is the height of a lighthouse from whose top the horizon is a distance of 24 miles?

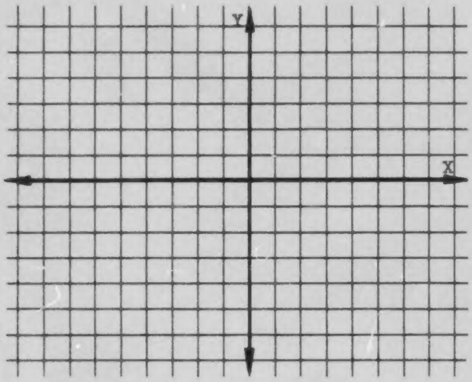
(c) What is the height of a balloon if the distance to the horizon is 48 miles?

(d) By what number was the height multiplied when the distance to the horizon was doubled in questions (b) and (c)?

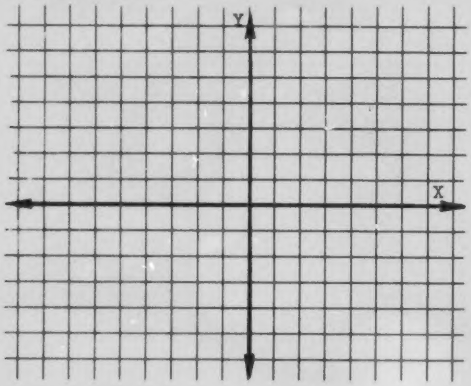
4. (a) Represent the identity function in arrow notation.

____

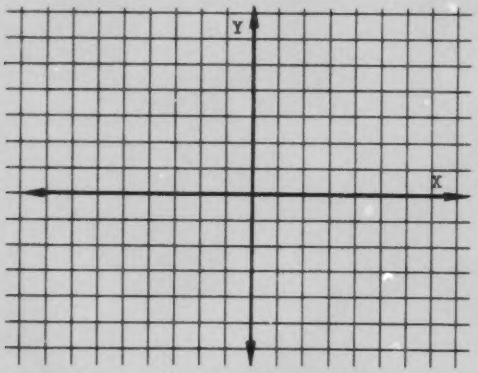
(b) Draw the graph of this function.



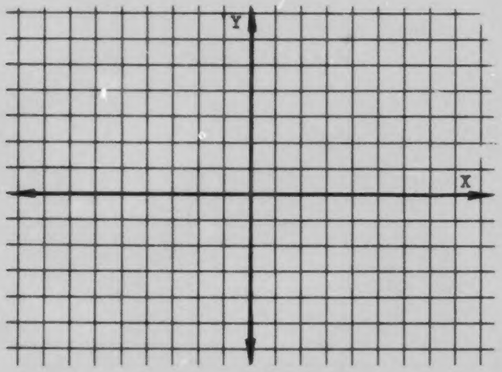
5. Draw the graph of the function, f, described by $f : x \to {}^-3$.



6. Draw the graph of the function, g, described by $g : x \rightarrow 3, \quad x \leq 0$.



7. Draw the graph of the function, h, described by $h : x \rightarrow 2x, \quad 0 \leq x \leq 2$.

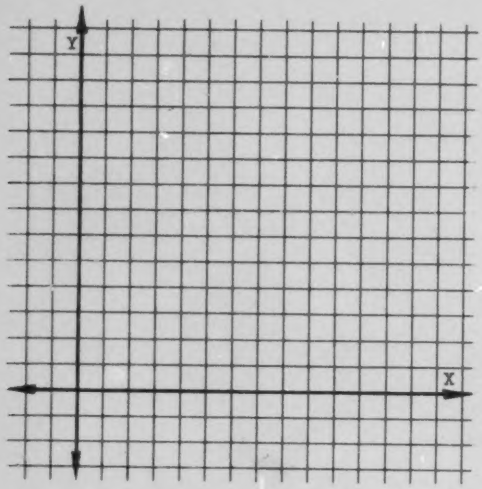


8. A function is expressed as $f : x \rightarrow 2x + 1, \quad x \geq 0$.

(a) Fill in the following table for this function.

Input: x	0	1	2	3	4	5
Output: 2x + 3						

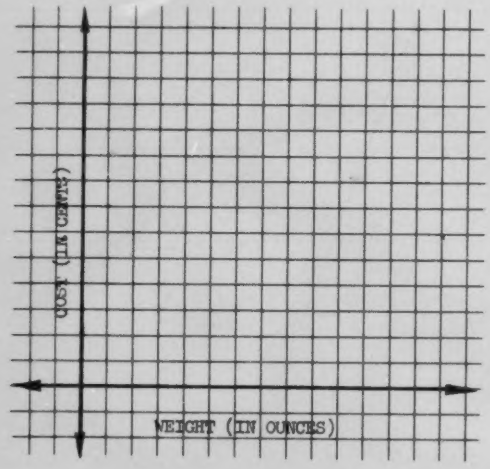
(b) Draw the ray that is the graph of this function.



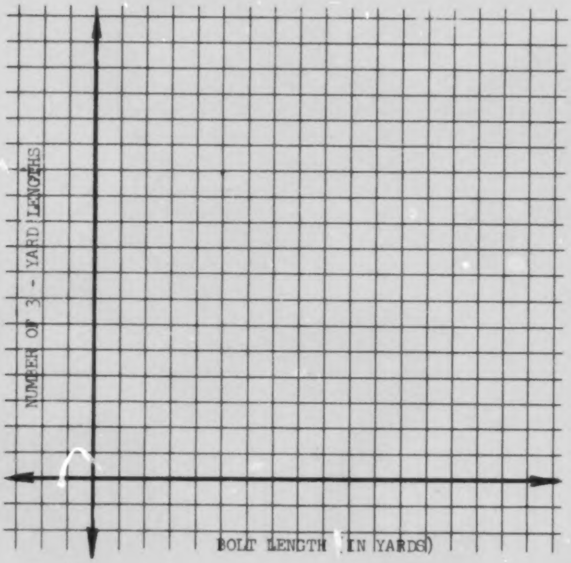
9. First class postage costs 6¢ per ounce. For each additional ounce, or part of an ounce, another 6¢ is charged.

(a) What kind of function is represented here? ____________________

(b) Draw a graph of this function for first class mail from 0 to 6 ounces inclusive.


COST (IN CENTS)
WEIGHT (IN OUNCES)

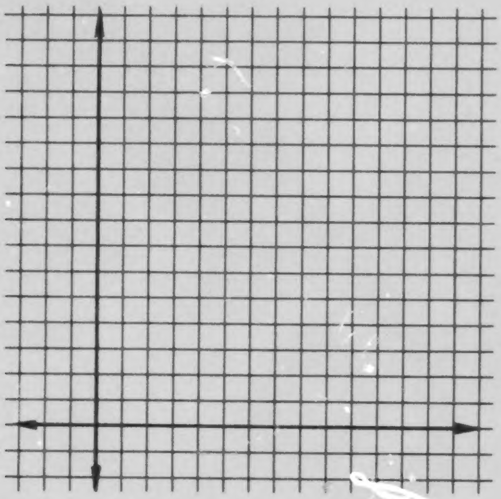
10. A dress manufacturer has several bolts of cloth of standard width. In order to make a certain type of dress it is necessary to cut three-yard lengths from these bolts. The correspondence between the length of a bolt and the number of three-yard lengths that may be cut from it is a function. The set of inputs consists of the bolt lengths and the set of outputs consists of the number of three-yard lengths obtained. Draw a graph of this function.



11. A circulating library charges 25 cents for the first 3 days and 5 cents for each additional day for renting a certain type of book.

(a) Represent this relationship as a function using the arrow notation.

(b) Draw a graph of this function.



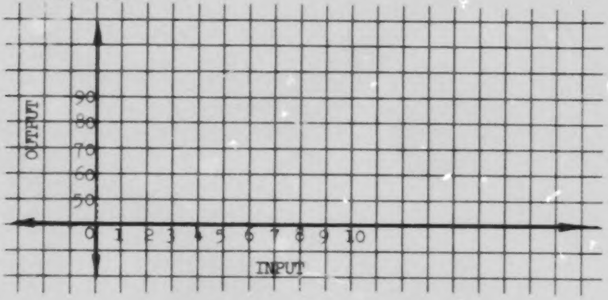
12. The table below shows the first ten scores (based on 100 points) earned by a student in a mathematics class.

Input: test number	1	2	3	4	5	6	7	8	9	10
Output: test score	90	80	85	70	80	75	85	60	90	85

(a) Express the "test score" function by using arrow notation.

(b) If we were to reverse the correspondence shown in the table, would we still have a function?

(c) On the coordinate system below, draw the graph of the "test score" function.



(d) The class average for these ten tests was 75. Draw the graph of $f : x \rightarrow 75, \quad x \geq 0,$ but use the same coordinate system on which you drew the "test score" function.

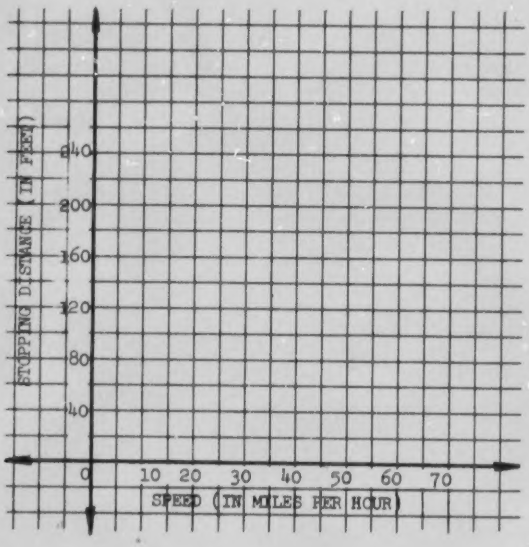
(e) Explain why it is not necessary to compute this student's average score in order to determine whether his average score was above or below the class average score.

(f) Was this student's average score above or below the class average?

13. Insurance companies are interested in letting people know the stopping distance of a car traveling at different speeds. If the input is the measure of the speed in miles per hour, and the output is the measure of the stopping distance in feet, the function is expressed by the following table:

Input: Speed	10	20	30	40	50	60
Output: Stopping Distance	16	42	78	124	180	246

(a) Plot these points on the coordinate system below, and draw a smooth curve through all these points.



(b) Is the graph of this function a line?

(c) According to your graph, what distance (approximately) is needed for a car to stop if:

its speed is 25 mph?

its speed is 37 mph?

its speed is 45 mph?

(d) Speeding automobiles often cause accidents. After such an accident, a policeman or insurance company representative may try to determine how fast a car was traveling by measuring the length of the skid marks. How fast may he assume that the car was traveling if the skid marks were measured to be 200 feet? 60 feet?

Answers to Suggested Test Items

Multiple Choice

1. E
2. C
3. B, C
4. C

Completion

1. Correspondence or association
2. Constant
3. (a) $P = 4s$
   (b) $f : s \rightarrow 4s$
   (c) $s > 0$
   (d) 36
   (e) $\frac{1}{2}$
4. (a) Yes
   (b) n (any letter will do)
   (c) $5n$
   (d) $f : n \rightarrow 5n$
   (e) 35
5.

Input:	0	1	2	3	4	5	6	7
Output: $3x + 2$	2	5	8	11	14	17	20	23

6. (a) $f : t \rightarrow 32t$
   (b) t
   (c) 320, 8
   (d) $t \geq 0$

7.

	Input	Output
(a)	$^-2$	$^-1$
(b)	$^-1$	$^-(\frac{1}{2})$
(c)	0	0
(d)	1	$\frac{1}{2}$
(e)	2	1

(f) $\frac{1}{2}n$

(g) $f : n \rightarrow \frac{1}{2}n$

8. (a) Step function

(b)

$0 \le a < 1$

$1 \le b < 2$

$1 \le c < 2$

Input	-3	-2.25	$-1\frac{1}{2}$	a	b	c	$4\frac{1}{4}$	5	$5\frac{1}{2}$
Output	-3	-3	-2	0	1	1	4	5	5

9. (a) Periodic

(b) 6.28 ... units (i.e., 2π units)

10. (a) Periodic

(b) 3 units

Problems

1. (a) not a function

(b) student → grade of student

(c) driver → drivers license number

(d) $(x,y) \rightarrow P$

(e) not a function

(a) child → father of child

(b) not a function

(c) drivers license number → driver

(d) P → (x,y)

(e) positive integer → non-negative integer

2. (a1) Yes   (a2) Yes

(b1) Yes   (b2) Yes

(c1) No   (c2) Yes

3. (a) $f : d \rightarrow \frac{2}{3}d^2,\ d > 0$

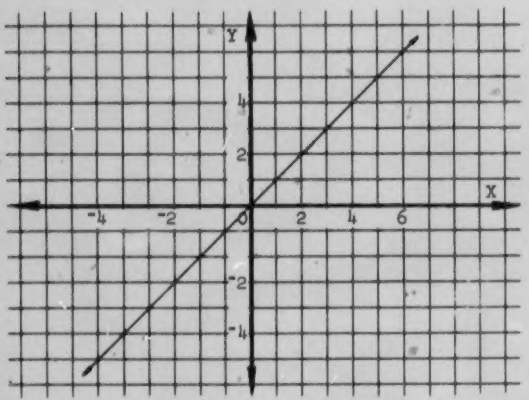
(b) 384 feet

(c) 1536 feet

(d) 4

4. (a) $I : x \rightarrow x$

(b)



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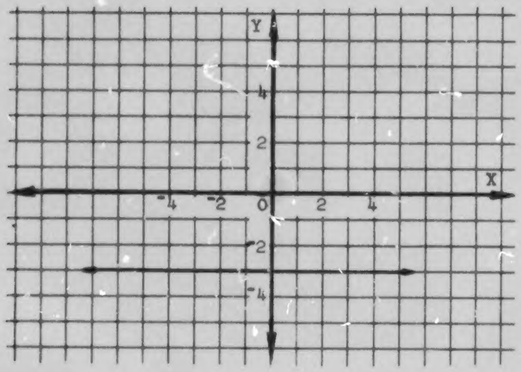

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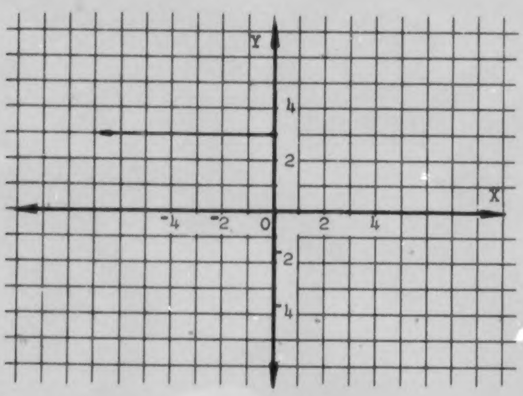

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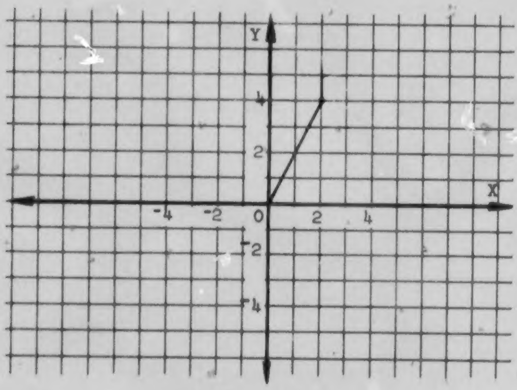
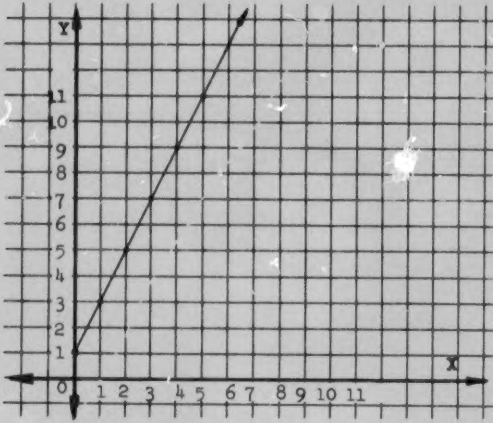


8. (a)

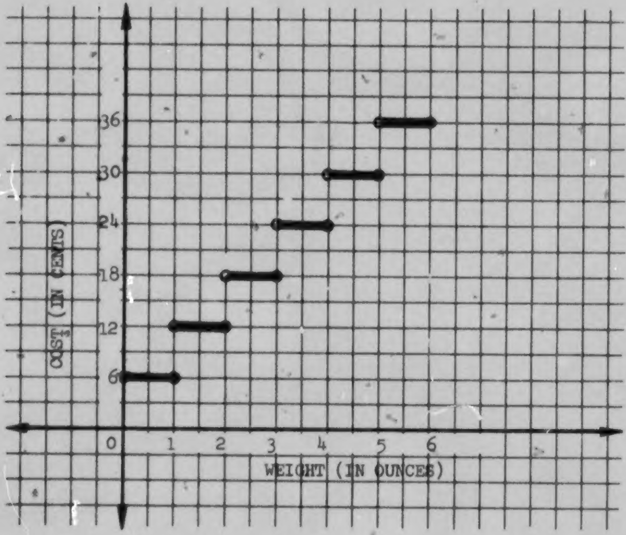
Input: x	0	1	2	3	4	5
Output: 2x + 1	1	3	5	7	9	11

(b)

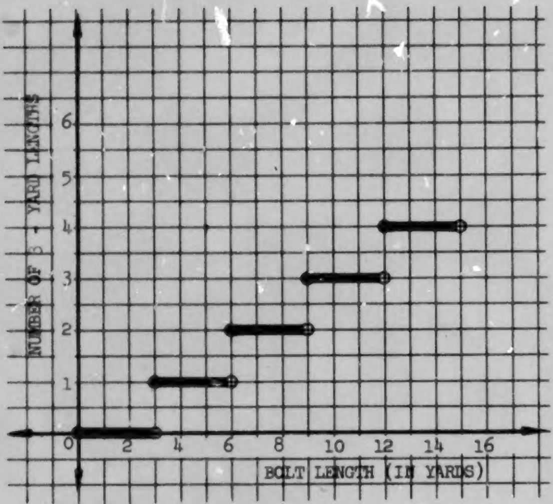


9. (a) A step function

(b)

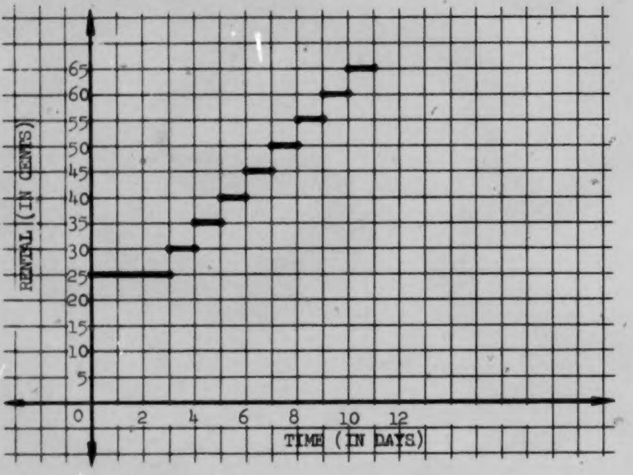


10.



11. (a) $f : n \to \begin{cases} 25 & \text{if } 0 < n \le 3. \\ 25 + 5(n - 3) & \text{if } n > 3. \end{cases}$

(b)



Note: It is possible to interpret the problem such that the arrow notation for the function is,

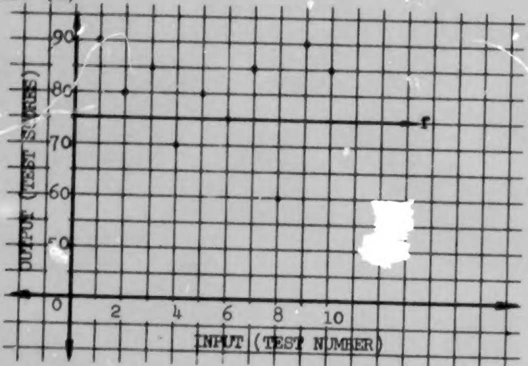
$$f : n \rightarrow \begin{cases} 25 & \text{if } n \text{ belongs to } \{1, 2, 3\} \\ 25 + 5(n - 3) & \text{if } n \text{ belongs to } \{4, 5, 6, \ldots\} \end{cases}$$

Of course, the graph now differs from that shown above.

12. (a) f : test number → test score

(b) No

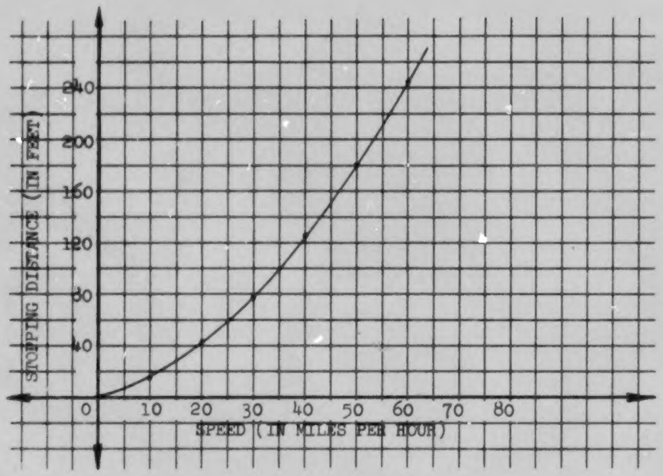
(c) and (d)



(e) We merely have to examine the graph of the "test score" function with reference to the graph of $f : x \rightarrow 75, \ x \geq 0$. We can determine, visually, how the sum of the distances of each point above f, to f, compares with the sum of the distances of each point below f, to f.

(f) Obviously, above the class average.

13. (a)



(b) No

(c) 25 → 60 (approximately)

37 → 105 (approximately)

45 → 150 (approximately)

(d) (approximately) 53 → 200

(approximately) 25 → 60

TEAR OUT SHEET

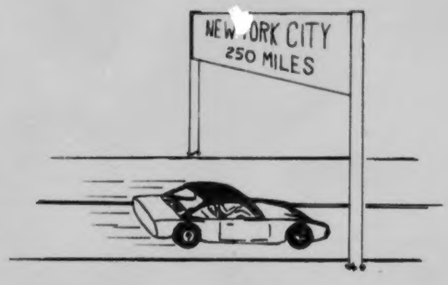
Exercises 2-1a (Class Discussion)

1. Complete the following table. Show your work below.

Distance traveled = 50 mi. per hr. × no. hrs. traveled

$$d = 50t$$

(time in hrs.) input t	(distance in miles) output $50t$
0	0
1	50
2	
3	
4	
5	250



Show work here:

$d = 50t$

If $t = 0$, then $d = 50(0) = 0$

If $t = 1$, then $d = 50(1) = 50$

If $t = 2$, then $d =$ ____________

If $t = 3$, then $d =$ ____________

If $t = 4$, then $d =$ ____________

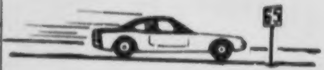
If $t = 5$, then $d = 50(5) = 250$

TEAR OUT SHEET

Exercises 2-1b (Class Discussion)

(time) t in hrs.	m/p/h	(distance) total mi.	average speed
t = 1	65	d = 65	65
t = 2	50	d = 115	$57 \frac{1}{2}$
t = 3	20	d = 135	____
t = 4	60	d = ____	____
t = 5	55	d = ____	50

1. Complete the work below, then complete the table to the left.



Find distance:

During 1st hr. he drove 65 miles.

During 2nd hr. he drove 50 mi. for a total distance of 65 + 50 = 115 mi.

During 3rd hr. he drove 20 mi. for a total distance of 115 + 20 = 135 mi.

During 4th hr. he drove 60 mi. for a total distance of ____________

During 5th hr. he drove 55 mi. for a total distance of ____________

Find average speed:

Average speed = $\frac{\text{distance}}{\text{time}}$

At the end of 1 hr., average speed = $\frac{65}{1}$ = 65 m/p/h.

At the end of 2 hr., average speed = $\frac{115}{2}$ = 57 m/p/h.

At the end of 3 hr., average speed = ____ = ____ m/p/h.

At the end of 4 hr., average speed = ____ = ____ m/p/h.

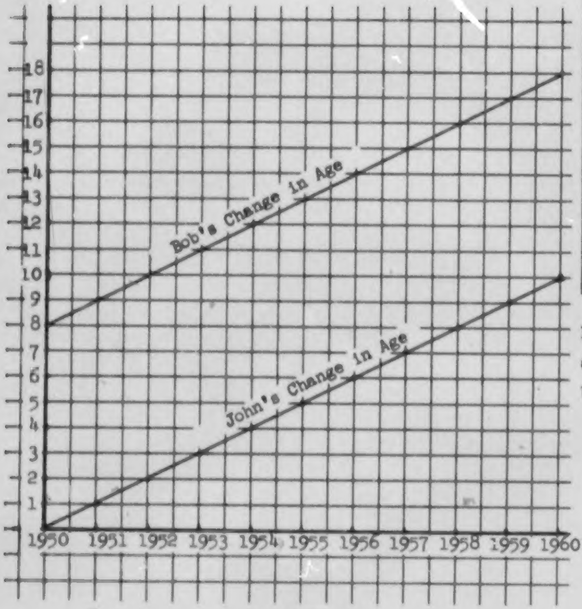
At the end of 5 hr., average speed = ____ = ____ m/p/h.

TEAR OUT SHEET

2. Fill in this table from the graph:

Year	Bob's age	John's age
1950	8	-
1951		1
1952		
1953		
1954		
1955		
1956		
1957		
1958		
1959		
1960		

From this table answer the questions below.



(a) Can Bob and John ever be the same age? ________

(b) Was Bob ever twice the age of John? ____ If so, when? ____

(c) Was Bob ever three times the age of John? ____ If so, when? ________

(d) Was Bob ever five times the age of John? ____ If so, when? ____

(e) Was Bob ever nine times the age of John? ____ If so, when? ____

Exercises 2-2

Follow the problems in your text, and fill in this table.

(no. seconds) t =	0	.5	1	1.5	2	2.5	3	3.5			
1. (no. ft. object falls) d =	0	4		36		100		196			
2. distance object falls each half-second		4									
3. difference between distance fallen each half-second											

4.

This space is for your work.

1. If $t = 0$, $d = 16(0)^2 = 16.0 = 0$
   If $t = .5$, $d = 16(.5)^2 = 16(.25) = 4$
   If $t = 1$, $d =$ ____________________
   If $t = 1.5$, $d = 16(1.5)^2 = 16(2.25) = 36$
   If $t = 2$, $d =$ ____________________
   If $t = 2.5$, $d = 16(2.5)^2 = 16(6.25) = 100$
   If $t = 3$, $d =$ ____________________
   If $t = 3.5$, $d = 16(3.5)^2 = 16(12.25) = 196$

2.

3.

4.

Chapter 3

INFORMAL ALGORITHMS AND FLOW CHARTS

3-1. Changing a Flat Tire

You have experienced many times the need to follow instructions in order to carry out some process successfully. For example, putting together a model airplane, following a recipe, playing a new game, complying with the rules of conduct set down by your parents, are all instances where it is necessary to follow instructions in order to carry out a process.

Definition. A list of instructions for carrying out some process step by step is called an algorithm.

Most processes can be represented as algorithms in many different ways. Here is one algorithm for changing a flat tire.

Algorithm for Changing a Flat Tire

1. Jack up the car.
2. Unscrew the lugs.
3. Remove the wheel.
4. Put on the spare.
5. Screw on the lugs.
6. Jack the car down.

You may feel that we have not put enough steps into our algorithm. We have not considered getting the equipment out of the trunk, placing the jack, removing the hub-caps, loosening the lugs before jacking up the car, etc. These are good objections. Still, our list is good enough for getting across the idea of an algorithm. When we get to the stage of writing algorithms for mathematical processes we will have to be much more precise.

A flow chart is a diagram for picturing an algorithm. We will give a flow chart for our flat tire algorithm and then explain it.

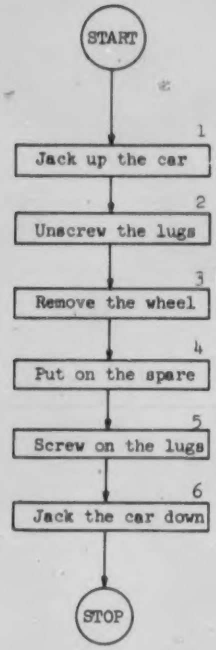

START
1
Jack up the car
2
Unscrew the lugs
3
Remove the wheel
4
Put on the spare
5
Screw on the lugs
6
Jack the car down
STOP

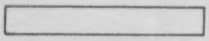
Figure 1. First flow chart for a flat tire.

In this flow chart, as in most, we see

and

.

We also observe in our flow chart that each instruction is enclosed in a frame or box. A little later on we will see that the shape of the frame tells us what kind of instruction appears inside. Commands to take some action are written in rectangular frames.

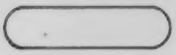


In Figure 1 all instructions are of this form, so they all have rectangular frames.

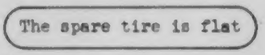
To carry out the process shown in a flow chart we go to START, follow the arrow to the first "box" and carry out the instruction given there, then follow the arrow to the next box, etc.

After drawing a flow chart we always look to see whether we can improve it. In the flat tire algorithm we forgot to check whether the spare was flat. Drivers seldom think at a service station to check the air in the spare tire, and sometimes it is flat when it is needed.

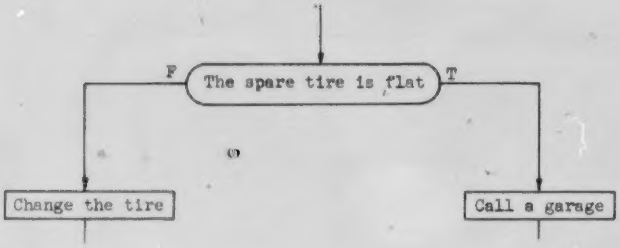
If the spare is flat then we certainly do not want to go to all the trouble of changing the tire. Instead we should call a garage. To make this decision we introduce a new kind of frame into our flow chart. The frame is oval in shape.



Inside this frame we find a statement on which we make a decision.



We have two exits from this box, one labeled T (true) and the other labeled F (false). After checking whether the statement is true or false, we leave the box at the corresponding exit and go on to the next box.



Such an oval box is called a decision box. When we put this flow chart fragment into our flat tire flow chart of Figure 1 we obtain the flow chart of Figure 2.

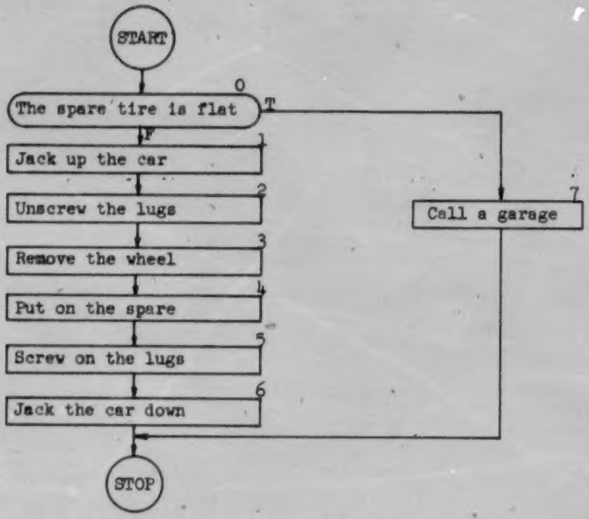
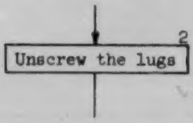
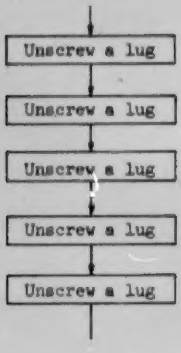

START
0
The spare tire is flat
T
F
1
Jack up the car
2
Unscrew the lugs
3
Remove the wheel
4
Put on the spare
5
Screw on the lugs
6
Jack the car down
7
Call a garage
STOP

Figure 2. Second flow chart for changing a flat tire.

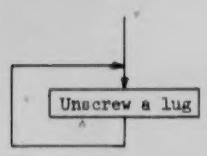
There is still one more improvement we would like to make on our flow chart. Let us look at box 2 in our flow chart.


2
Unscrew the lugs

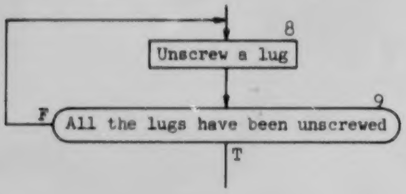
Actually this stands for a number of tasks, or rather the repeating of the same task. Since this automobile wheel has five lugs, one way of showing this is to have five frames.


Unscrew a lug
Unscrew a lug
Unscrew a lug
Unscrew a lug
Unscrew a lug

This is correct but we can simplify this diagram by introducing a loop.


Unscrew a lug

We see that when we leave this box we are sent right back to repeat the task. The trouble with this idea as that we have no way of getting out of the loop to the next task. We are caught in an endless loop. We can correct this situation by placing a decision box in our flow chart as shown in the figure below.


8
Unscrew a lug
9
F
All the lugs have been unscrewed
T

We get our final flow chart for changing a flat tire (Figure 3) by replacing box 2 by boxes 8 and 9 and making a similar replacement for box 5.

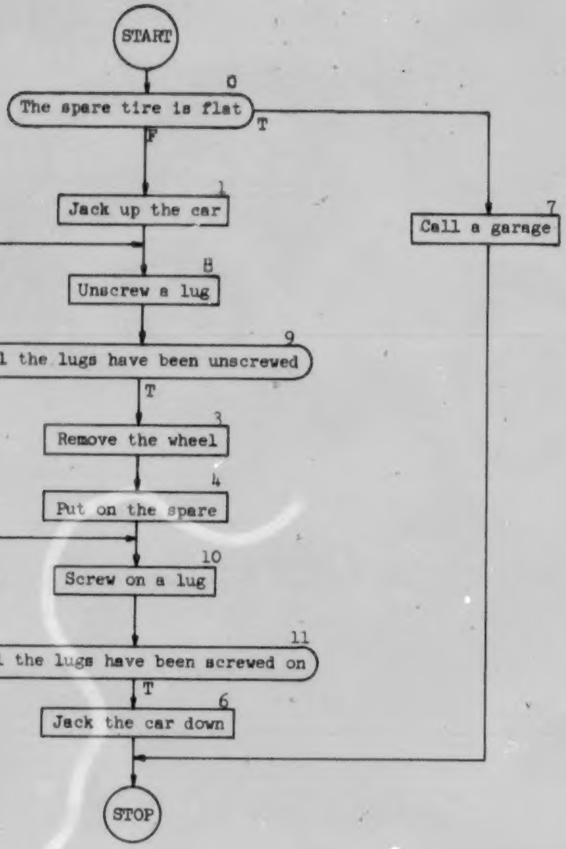

START
0
The spare tire is flat
T
F
1
Jack up the car
7
Call a garage
8
Unscrew a lug
9
F
All the lugs have been unscrewed
T
3
Remove the wheel
4
Put on the spare
10
Screw on a lug
11
F
All the lugs have been screwed on
T
6
Jack the car down
STOP

Figure 3

Exercises 3-1

(Class Discussion)

1. Mark's father's favorite breakfast consisted of orange juice, toast, milk, and 1 fried egg (sunnyside up) basted in butter. On Fathers' Day, Mark decided he would surprise his father by cooking breakfast for him but didn't know how to fry the egg. Mark's older sister (who was a computer programmer) constructed a flow chart showing Mark how to fry an egg the way their father liked it cooked.

To start, Mark's sister listed the basic processes needed to cook an egg.

1. Place frying pan on burner.
2. Set heat under frying pan to medium-low.
3. Put butter in pan.
4. Break egg into frying pan.
5. Baste egg with melted butter.
6. Serve egg to Dad.

(a) Construct the first flow chart for frying an egg. (See Figure 1, page 2.)

(b) It is always possible that the family may be out of butter or eggs. Change your flow chart to account for this possibility by inserting a decision box in the correct place with the statement "We have both butter and eggs."

(c) As there are two exits from a decision box, one labeled T (true) and one labeled F (false), the sister decided that if they were out of butter or eggs Mark's father would be served cold cereal. Change your flow chart to take account of this possibility.

(d) Before breaking an egg into the frying pan the butter should be melted. Insert a decision box with the statement, "the butter has melted", into the flow chart.

(e) If it is T (true) that the butter has melted, the next step would be to "break the egg into the frying pan". If the statement is F (false) then Mark must "wait $\frac{1}{2}$ minute". Using a loop and a rectangular box, change your flow chart to account for this possibility.

(f) To determine whether the egg is cooked or not, the white should not be transparent. If it is transparent, then more basting is needed. Change your flow chart by putting in a decision box, and loop, with the statement "the egg white is transparent".

With this last change, we get our final flow chart for frying an egg.

3-2. Algorithms, Flow Charts, and Computers

First you should know that our study of algorithms and flow charts is inspired by computers. Basically a computer does arithmetic: it can add, it can subtract, it can multiply, it can divide. The computer can perform these operations very rapidly. In fact, the computer can do millions of these arithmetic operations in a single second. By combining a vast number of simple arithmetical calculations a computer can solve in less than a minute a problem which might require weeks of hand computations. In an hour it can handle a problem needing years of hand computations.

The computer's great speed in performing an arithmetic operation would be of little value if after each of these calculations we had to stop and give the instruction as to what to do next. The time required to give the computer its millions of instructions would make it impossible for the computer to reduce computing time by much more than half. The secret is that all the instructions are put into the computer at the beginning of the problem so that the computer can get at these instructions using the same kind of speed which it uses on its arithmetic operations.

Still, if the computer is to perform millions of operations, will this require millions of instructions? Surely these would require an enormous amount of time to prepare. The answer is that as few as ten or twenty actual instructions may be used to tell the computer to make millions of calculations. The secret here lies in algorithms involving repetition. On a small scale we have met this idea in our flat tire algorithm in the preceding section when we introduced the loop in the flow chart. We passed through this loop several times before leaving it to perform another task. The instructions given a computer are in the form of algorithms, involving much looping, so that a small number of instructions can result in a large number of operations.

The first task in preparing a problem for a computer is to construct an algorithm for the problem. Usually it is not possible to see in advance just exactly what the computer will do at each step. We do know that if the instructions are followed the correct answer will be obtained. It may seem difficult to understand how we can give instructions for solving a problem but not know in advance what steps will be taken. The following exercises will illustrate this.

Exercises 3-2a

(Class Discussion)

1. Suppose a hiker is lost in the woods without a map or a compass. We will assume for simplicity that the man has found a river or stream. We want to construct an algorithm that will help the hiker find his way back to civilization so that he does not wander around aimlessly and never find a town. (We will assume that all rivers flow toward the sea.)

   (a) After the hiker has found a river what is the first command that you would give him?

   (b) If the hiker follows the river and comes to a town, then what command would you give him in the algorithm?

   (c) What would you instruct the hiker to do if the river flows into another river?

   (d) What would you instruct the hiker to do if the river flows into a lake?

   (e) What would you instruct the hiker to do if he comes to a river that flows out of the lake?

   (f) What would you instruct the hiker to do if the river flows into the sea?

   (g) Compare your answers with the following flow chart.

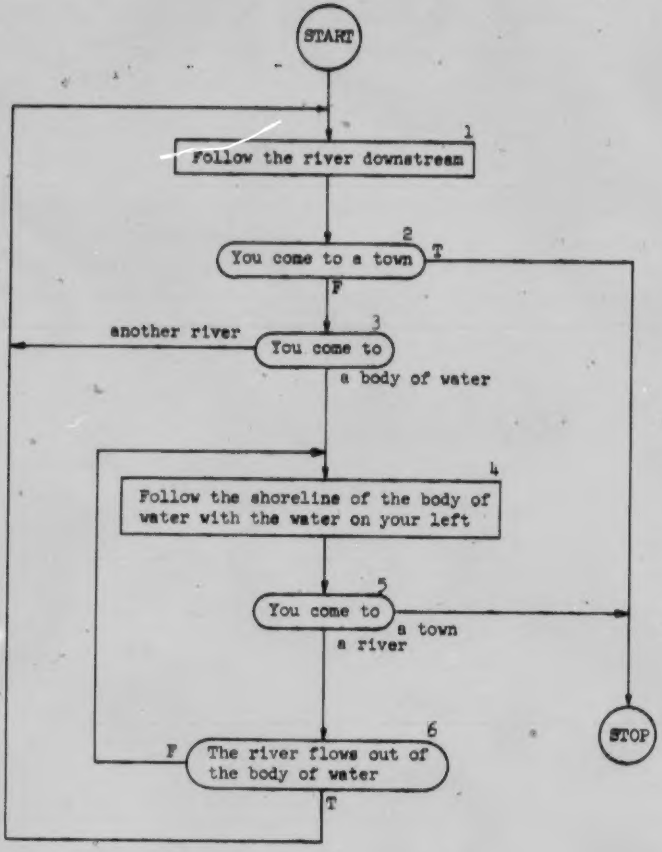

START
1
Follow the river downstream
2
You come to a town
T
F
3
another river
You come to
a body of water
4
Follow the shoreline of the body of water with the water on your left
5
You come to
a town
a river
6
F
The river flows out of the body of water
T
STOP

Figure 4. Lost in the woods.

As you can see, even though we have no idea what path the lost hiker will actually travel in following our flow chart, we believe that this flow chart will finally lead him back to civilization.

Of course, there are many different algorithms that could be constructed which would also lead the hiker back to civilization. Notice that in two of our decision boxes the exits are labeled with words rather than T or F. Also notice that the "body of water" referred to can be either the sea or a lake.

Exercises 3-2b

1. Two parties of hunters (A, B, C, D in one party, and X, Y in the other) on Gull Island became separated during the hurricane of 1964. Their positions after the storm are shown on the map in Figure 5. Each followed the flow chart of Figure 4 to find his way back to civilization.

   (a) For each hunter give the town he finally reached.

   (b) List the hunters in the order of the distance they traveled.

   (c) Which hunters from different parties reached the same town?

   (d) Which hunters from the same party arrived at towns that are the farthest apart?

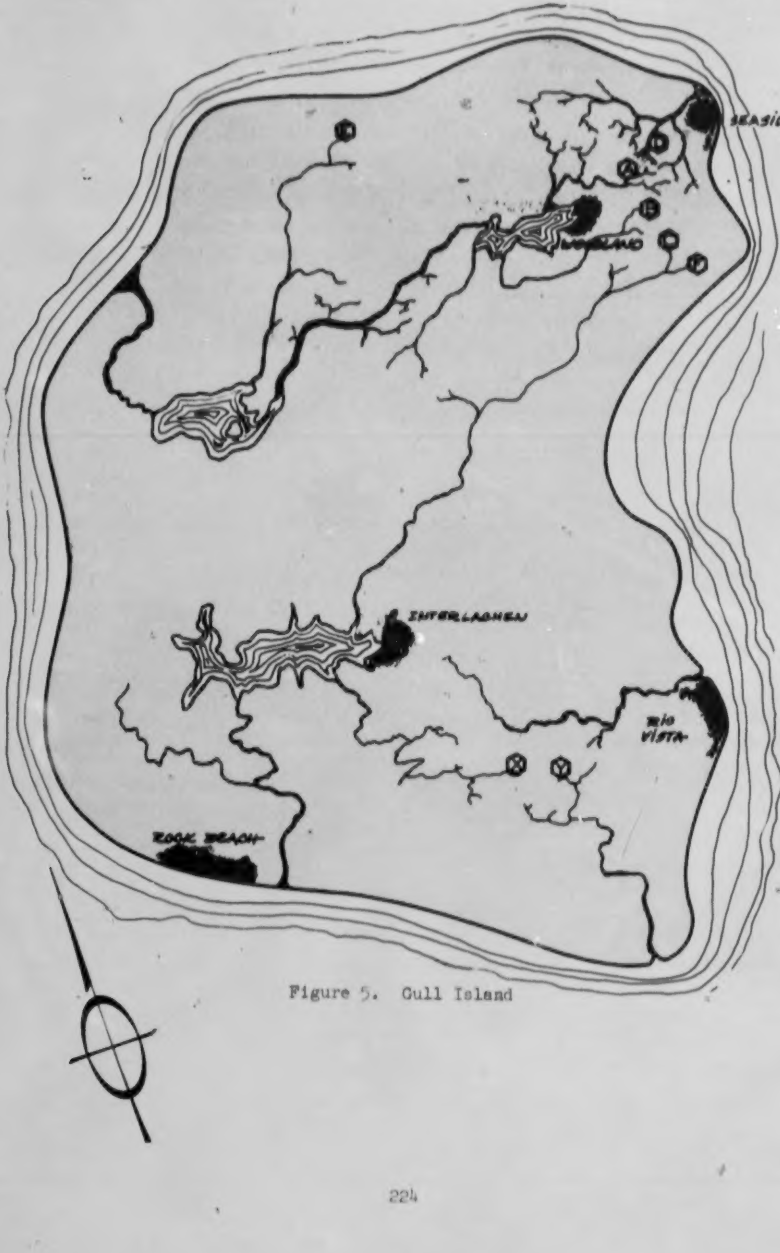

SEASIDE
HIGHLAND
INTERLACHEN
RIO VISTA
ROCK BEACH

Figure 5. Gull Island

2. In 1768 Gull Island was a haven for pirates, and it is claimed by some that buried treasure still exists somewhere on the island. An old pirates' log book was found recently which indicated that the treasure was buried under an old oak tree at the headwaters of some stream on Gull Island. It is known that there are old oak trees at points A, B, C, D, E, F, X, and Y whose positions are shown on the map on the preceding page. The instructions given in the log book are shown in the following flow chart, Figure 6. Find the point where the treasure hunters should dig for the treasure.

____

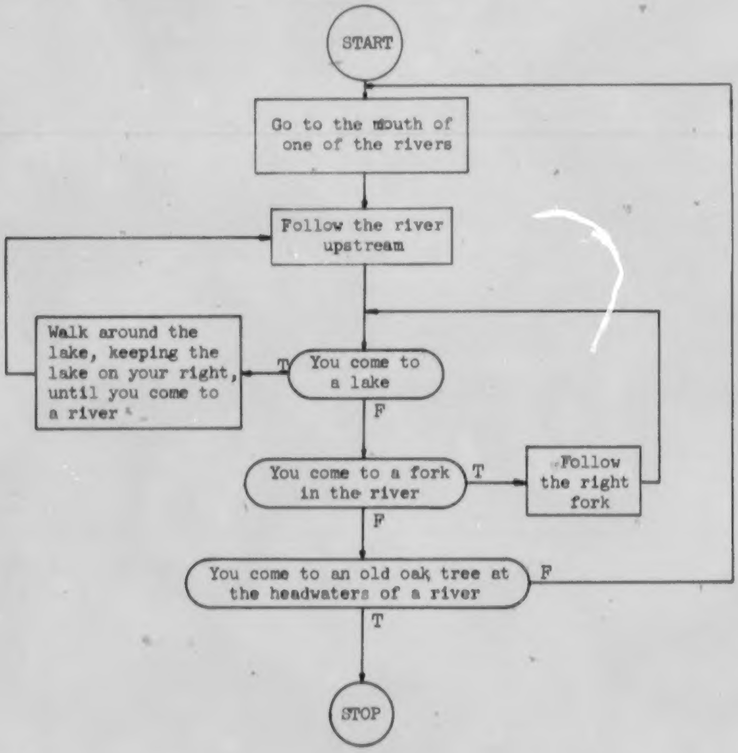

START
Go to the mouth of one of the rivers
Follow the river upstream
You come to a lake
T
Walk around the lake, keeping the lake on your right, until you come to a river
F
You come to a fork in the river
T
Follow the right fork
F
You come to an old oak tree at the headwaters of a river
F
T
STOP

Figure 6

3-3. Assignment and Variables

In computing work a *variable* is a letter used to represent a number. You have seen examples of letters used in this way in such formulas as

$$A = L \cdot W,$$

where L and W stand for the length and width of a rectangle and A represents its area. As another example we have the formula

$$W = R \cdot T$$

for computing wages. Here, R stands for the hourly rate of pay of a worker in dollars, T for the time worked in hours, and W for his wages in dollars.

In computing work, at any particular time, a variable must represent one definite number. This number is called the *value* of the variable. Although at any particular time each variable has just one definite value, the value may change from time to time. For example, we might wish to use the formula, $W = R \cdot T$, to compute the wages of several workers who may have different pay rates or may work for different periods of time. Soon we will develop a flow chart for doing just that.

Before drawing this flow chart we will devise a *model* which will show very clearly how variables are used in computing.

We imagine that for each variable used in our problem there is an associated *window box*. On top of each box is engraved the associated variable. Inside the box is a slip of paper with the *present value* (or *current value*) of the variable written on it. The variable is a name for the number that appears inside.

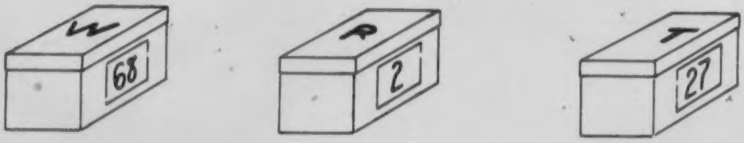


Figure 7. Memory

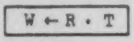
Each box has a lid which may be opened when we wish to assign a new value to the variable. Each box has a window in the side so that we may read the value of the variable without changing the value. These window boxes make up the *memory* of our computer.

We imagine that the computing operation is performed by a "master computer" and two assistants called the "assigner" and the "reader". (In a real computer their tasks are performed by electrical circuits.) The master computer receives his instructions from a flow chart and gives certain tasks to the assistants.

Suppose we wish to have a worker's wages computed using the formula

$$W = R \cdot T.$$

The instruction to compute the value of W will come to the master computer in the following flow chart box.



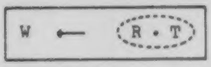
Inside this box we find an assignment statement. To read this statement aloud we say,

"Assign to W the value of R · T."

The left pointing arrow is called the assignment operator. This arrow is to be thought of as an order or a command. Rectangular flow chart boxes will always contain assignment statements. Such a rectangular box is therefore called an assignment box.

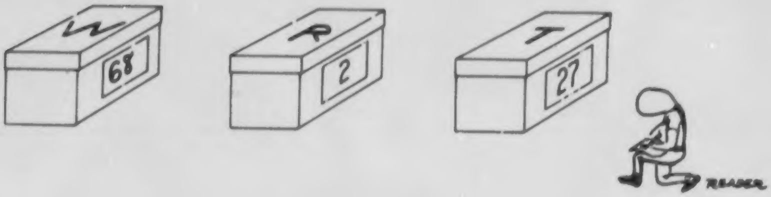
Next, we shall see what happens when the master computer receives such an instruction. We shall assume that R and T (but not W) already have the desired values, say those shown in Figure 7. (How they obtained these values will be discussed later on.)

The computation called for in the assignment statement occurs on the right-hand side of the arrow, so the master computer looks there first.

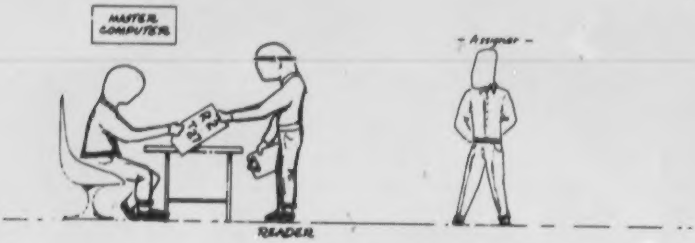


He sees that he must know the values of R and T. So, he calls the reader and sends him out to bring these values from the memory.

The reader goes to the memory and finds the window boxes labeled R and T. He reads the values of these variables through the windows,



writes the values down, and takes them back to the master computer.



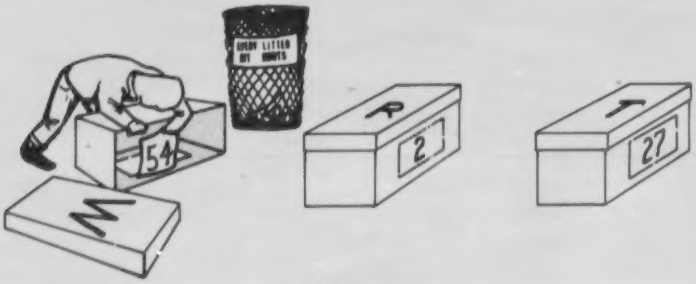
The master computer computes the value of R • T using the values of R and T brought to him by the reader. He gets the value 54 for R · T.

Now the master computer looks at the left-hand side of the arrow in his instruction.

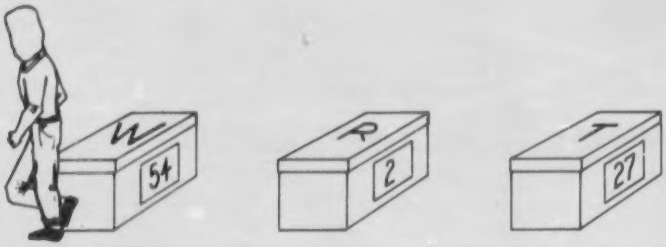
$$\boxed{W \quad \longleftarrow \quad R \cdot T}$$

He sees that he must assign the computed value of R • T, namely 54, to the variable W. So he writes "54" on a slip of paper, calls the assigner, and tells him to assign this value to the variable W.

The assigner goes to the memory, finds the window box labeled W, and dumps out its contents.



Then he puts the slip of paper with the new value in the box, closes the lid, and returns to the master computer for a new task.



We say that assignment is destructive because it destroys the former value of the variable. Reading is nondestructive because this process in no way changes the values of any of the variables in the memory.

Check Your Reading

1. In computing work what is a variable?

2. What will you always find inside an assignment box?

3. The left pointing arrow is to be thought of as an ______ or a ______.

4. Why do we say assignment is destructive?

5. Why do we say reading is nondestructive?

Exercises 3-3a

(Class Discussion)

1. (a) The starting (or initial) values of B and C are given in the table below. Fill in the values of these variables after carrying out the instruction in the assignment box on the right.

	B	C
starting values	9	4
final values		

$\boxed{B \longleftarrow C}$

(b) Instructions the same as in part (a).

	B	C
starting values	9	4
final values		

$\boxed{C \longleftarrow B}$

(c) If we compare the values of B and C after either of the assignment statements

$$B \longleftarrow C \qquad \text{or} \qquad C \longleftarrow B,$$

what do we find?

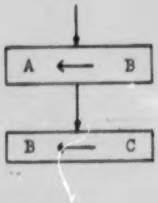
(d) Are the effects of the assignment statements

$$B \longleftarrow C \qquad \text{and} \qquad C \longleftarrow B$$

the same or different? Why?

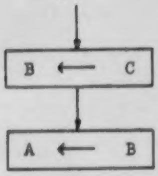
2. (a) The starting values of A, B, and C are given in the table. The two assignments on the right are to be performed in the indicated order. Fill in the values in the table.

	A	B	C
starting values	9	11	13
values after first assignment			
final values			



(b) Instructions the same as in part (a).

	A	B	C
starting values	9	11	13
values after first assignment			
final values			



(c) In what way are the instructions in the assignment boxes of parts (a) and (b) the same?

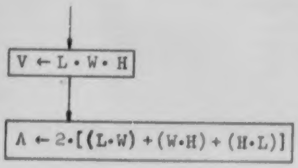
(d) In what way are they different?

(e) Does the order in which two assignment statements are carried out affect the final result?

Exercises 3-3b

Compute the values of V and A according to the two assignments on the right for each set of values of L, W, and H given in the table below.

	L	W	H	V	A
1.	7	3	2		
2.	8	3.5	5		
3.	11	9	7		
4.	12	2.3	4.6		

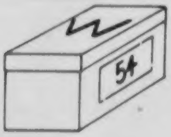


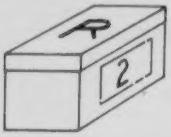
3-4. Input and Output

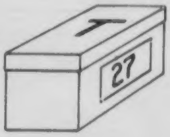
In the previous section the master computer was instructed to perform the following assignment:

$$\boxed{W \longleftarrow R \cdot T}$$

The values of R and T at the time were 2 and 27. The task was performed with the result that the memory looked like this:

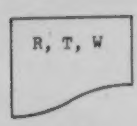






The desired value of W is now stored in the computer's memory. Next we want the computer to produce the answer so we can see it. This will require an instruction to the master computer to print out or output the answer. While we are about it we may as well have the computer print the values of R and T along with the value of W. In this way, in case we have to compute the wages for several workers we will know which wages go with which values of R and T.

Our instruction to the master computer to output the values of R, T, and W takes this form:



We see that we have a new shape of flow chart box. Inside the box, separated by commas, are the variables whose values we wish to know. This box is called an output box. When the master computer receives this instruction he sends the reader to bring to him the values of these variables. When the reader returns with these values the master computer types them out for us to see in the same order as they are listed in the output box.

The shape chosen for our output box suggests a page torn off a line printer, one of the most common computer output devices.

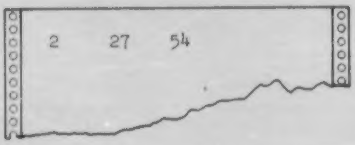
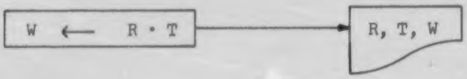


Figure 8. Output from line printer

In Figure 8 we see how the output data for our problem might look if printed by a line printer.

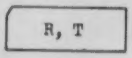
Putting our two flow chart boxes together in the proper order we now have:



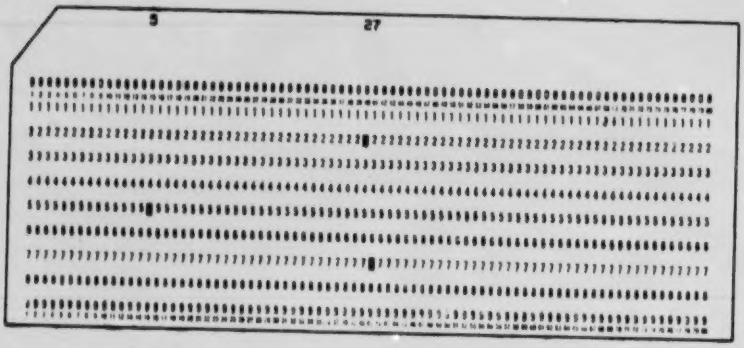
We notice that the actual numbers, the values of the variables being output, do not appear in our flow chart.

Now we give our attention to how the variables R and T get their values. Remember that we are making a flow chart for finding the wages of a worker whose hourly rate of pay and hours worked are given. The rate of pay and hours worked must be read in as input from outside the machine. Just as the actual output values of our variables do not appear in our flow chart, neither do the input values of our variables.

Instead, the master computer will be given an instruction to take whatever values are supplied from outside the machine and assign these values to R and T. This instruction will take the form:

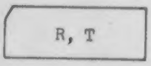


The shape of this box is supposed to suggest a punched data card of the type shown below.



You may have seen such cards and referred to them as "IBM cards". We will call them punch cards or "Hollerith cards" after Herman Hollerith who invented them in the late 19th century. The holes punched in the card are a special code for the numbers printed directly above them.

When we see an input box like



we know there will be a stack of punch cards, each card having two numbers printed on it. When the master computer receives the instruction

$\boxed{R, T}$ he reads the first card in the stack. He then gives the values printed on this card to the assigner to be assigned respectively to R and T. The card which was read is removed from the stack.

Before proceeding to the next part of flow charting, let us review the ideas studied thus far. We have discussed four kinds of flow chart boxes.

The assignment box. This box is rectangular in shape and always contains an arrow pointing left.

$$\boxed{W \leftarrow R \cdot T}$$

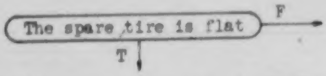
On the left-hand side of this arrow we always find a single variable. On the right-hand side of the arrow we find an arithmetic expression. The assignment box is a command to:

(1) read from the computer's memory (window boxes) the values of any variables occurring to the right of the arrow;

(2) using these values for the variables, compute the value of the expression on the right of the arrow;

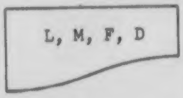
(3) assign this value to the variable on the left of the arrow (that is, put this value in the associated window box).

The decision box. This box is oval in shape and always has two exits, one labeled T (true) and the other labeled F (false).



The decision box always contains a statement instead of an instruction. After checking whether the statement is true or false, we leave the box at the corresponding exit and go on to the next activity.

The output box. This box is shaped like a sheet of paper torn off a line printer.

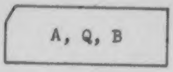


Inside the box we find a single variable or a list of variables separated by commas. No computation takes place in an output step. The output box is a command to:

(1) read the value of each listed variable from its window box;

(2) print out these values in the order listed.

The input box. This box is shaped like a punch card.



Inside the box we find a single variable or a list of variables separated by commas. No computation takes place in an input step. The input box is a command to:

(1) read, for each listed variable, a value supplied from outside the computer;

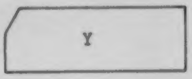
(2) assign these values in order to the variables in the list.

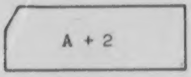
We note that assignment is called for in an input box as well as in an assignment box. The difference is that: in an assignment box, the assigned values are obtained from calculations done inside the computer using values obtained from inside the computer; but in an input box, no computation is involved and the values come from outside the computer.

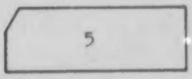
Exercises 3-4a

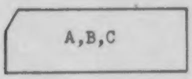
(Class Discussion)

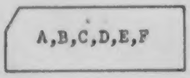
1. Which of the following are valid input boxes? If not valid, tell why.

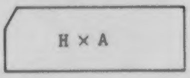
(a)


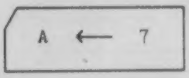
(b)


(c)


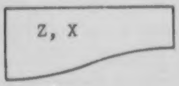
(d)


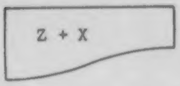
(e)


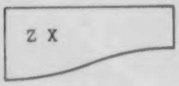
(f)


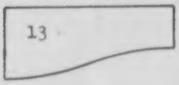
(g)


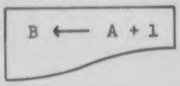
2. Which of the following are valid output boxes? If not valid, tell why.

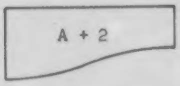
(a)


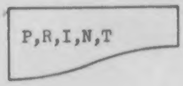
(b)


(c)


(d)


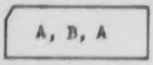
(e)


(f)


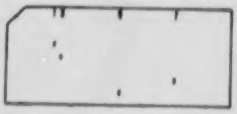
(g)


21

3. (a) What do you think would happen if we had an input box like this:

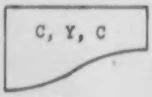


(b) Suppose that the punch card to be read is:



What values would be in the memory after the card has been read?

4. (a) What do you suppose that the computer would print out for an output box like this:



(b) Suppose that the computer memory contains the values 19 and 11 for C and Y, respectively. What would the computer print out?

5. Which of the following are valid assignment boxes? If not valid, tell why.

(a) $\boxed{A,\ B,\ C}$

(b) $\boxed{A \leftarrow B + C \times D}$

(c) $\boxed{A + C \rightarrow B}$

(d) $\boxed{A \leftarrow 2}$

(e) $\boxed{2 \leftarrow A}$

(f) $\boxed{2 \leftarrow 1 + 1}$

(g) $\boxed{2 \leftarrow 2}$

(h) $\boxed{A + B + C}$

(i) $\boxed{A \leftarrow B,\ C}$

(j) $\boxed{A \leftarrow B \text{ or } C}$

(k) $\boxed{A = L \times W}$

(ℓ) $\boxed{V \leftarrow E \times E \times E}$

Now, let us return to the flow chart for the computation of wages.

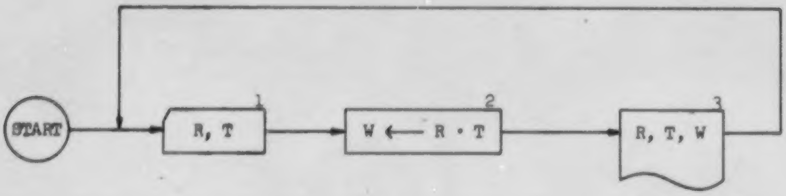
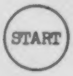
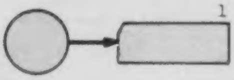
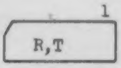
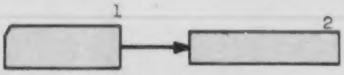
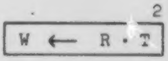
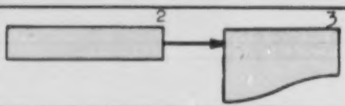
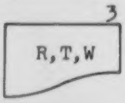
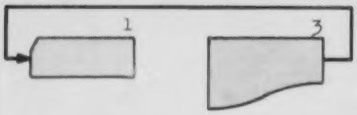
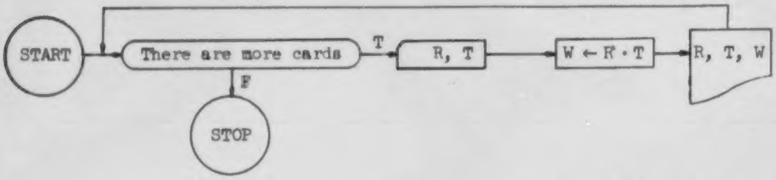


Figure 9. Flow chart for computing wages.

The steps involved in this flow chart, together with the flow chart boxes which call for these actions, are shown on the following page.

STEP	ACTION	CORRESPONDING FLOW CHART BOXES
1	Start.	
2	Go to box 1.	
3	Input two numbers and assign them to R and T.	
4	Go to box 2.	
5	Compute the value of $R \cdot T$ and assign this value to W.	
6	Go to box 3.	
7	Read the values of R, T, and W and output these values.	
8	Return to box 1 and repeat the process with a new set of data values.	

You may have noticed that the flow chart, in Figure 9, provides no instruction to stop. Without such an instruction the flow chart would suggest an "endless loop". We could introduce a decision box before box 1 so that our flow chart would look like this:



We will not ordinarily do this. The reason is that one of the jobs of the input box is to stop the computing process when there are no more cards to be read. Most computers require a special card at the end of a data deck which tells the computer to stop. However, we will agree that

> If a flow chart arrow carries us into an input box and it turns out that there are no cards left in the stack, then the computation is to stop.

To help you really understand assignment and variables we urge you to act out the operations of a simple computer that are described in the following Class Activity Exercises.

Exercises 3-4b

(Class Activity)

A Simple Computer

The Parts: The Master Computer, The Assigner, and The Reader.

Materials: Three window boxes (shoe boxes with holes cut in the side will do), blackboard, chalk, pencils and two pads of paper.

To prepare for the operation of the computer:

(1) Mark the top of one window box with the letter R, another T, and a third W.

(2) Put the following inputs on a deck of six cards and put the output headings on the blackboard.

INPUT		OUTPUT		
R	T	R	T	W
2.00	27			
2.50	38			
1.75	36			
2.10	40			
2.25	39			
1.95	40			

Operating the computer:

(1) Each line in the input list represents a punched card.

(2) The Master Computer should work through the flow chart of Figure 9.

(3) After starting, the Master Computer should read box #1 which tells him to input values of R and T. He tells the Assigner, "Pick up the first data card and put the value of R in the R box and the value of T in the T box."

(4) The Master Computer then reads box #2. He says to the Reader, "Go read the values of R and T, write them down, and bring them back to me."

(5) The Master Computer then computes the value of $R \cdot T$. He then says to the Assigner, "Take this value of $R \cdot T$ and put it in the W box."

(6) The Master Computer reads box #3 next. He says to the Reader, "Go to the R, T, and W boxes, write down these values, and bring them back to me."

(7) The Master Computer then tells the Assigner to write these values on the board under the appropriate Output headings.

(8) Repeat steps (3) through (8) until all data cards are used.

To more closely parallel the operation of a computer the Reader and the Assigner should really perform only one task at a time. One can observe that "reading the value of the variable" is nondestructive and that "assignment" is destructive.

Also as the "simple computer" operates it is helpful to observe the role of the variable in computing work. One can see that the variable represents, at any particular time, one definite number, and that the value of the variable may change from time to time.

Exercises 3-4c

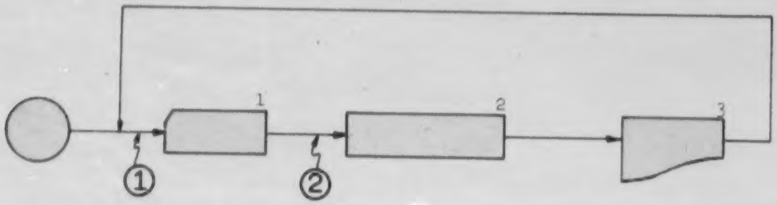
1. The flow chart of Figure 9 is to be carried out with four sets of values of R and T (four punch cards). The values on these cards are found in the table below:

	R	T
First card	2	27
Second card	2.15	39
Third card	1.87	41.75
Fourth card	1.945	37.25

(a) Display the output using one card for each time through the output box 3.

(b) Same as part (a) but this time round off the wages to the nearest penny. (R is given in dollars per hour and T in hours.)

2. The flow chart of Figure 9 is shown in silhouette.



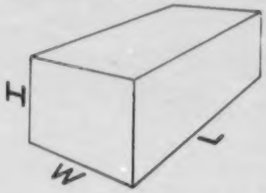
Using the data shown in problem 1, find the values of the variables R, T, and W at each of the following stages:

(a) the second time we arrive at the point marked ①,
(b) the third time we arrive at ②,
(c) the last time we arrive at ②,
(d) the first time we arrive at ②,
(e) the first time we arrive at ①.

[Note: In some parts of this question you will be unable to give the values of some of the variables. When this happens indicate which variables do not have their values determined by the available information.]

3. The volume, V, of a box is given by the formula

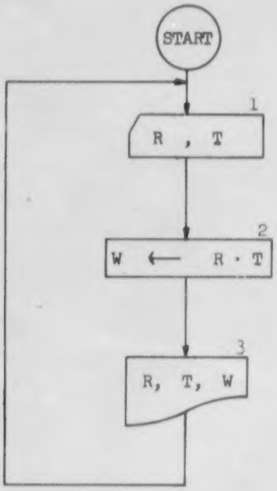
$V = L \cdot W \cdot H.$



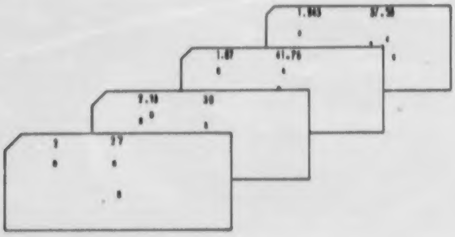
Draw a flow chart for inputting various values of L, W, and H, computing the values of V, and outputting the computed values.

3-5. Using a Variable as a Counter

In the last section we built a flow chart for computing the wages of employees. If we draw this flow chart vertically instead of horizontally it looks like this:



If the input data are the following:



then the output values look like this:

```
2          27          54
2.15       39          83.85
1.87       41.75       78.0725
1.945      37.25       72.45125
```

We might wish to have our lines of output numbered for easy reference so as to appear as follows:

1	2	27	54
2	2.15	39	83.85
3	1.87	41.75	78.0725
4	1.945	37.25	72.45125

In order to number the lines of output we put an extra variable, which we will call N, in our output box.

N, R, T, W

We would like to make the variable N take on the values 1, 2, 3, ..., in order. To do this numbering we place in our flow chart an additional assignment box.

$N \longleftarrow N + 1$

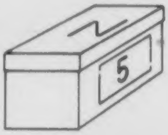
To see what this instruction means we remember that an assignment statement is a command to:

first, look up the values of the variables on the right;

second, using these values compute the value of the expression on the right;

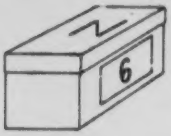
third, assign this computed value to the variable on the left.

To see how this works out with the instruction, $N \leftarrow N + 1$, suppose that the value of N is 5 before carrying out the instruction. We look up the value of N,



Window box before

which is 5; we compute N + 1, which is 6; we assign this value to N.



Window box **after**

The effect of this instruction, then, is to increase the value of N by 1. This is just what we wanted. So we place this box in our flow chart as follows:

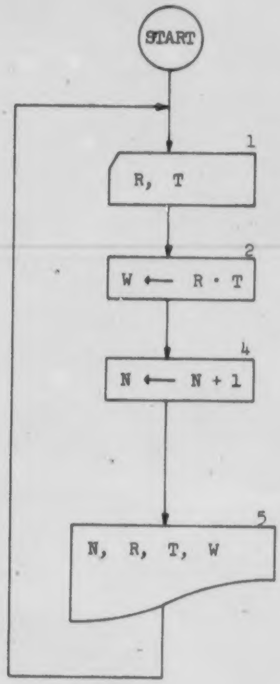
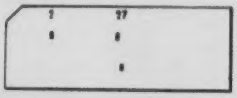

START
1
R, T
2
W ⟵ R · T
4
N ⟵ N + 1
5
N, R, T, W

Figure 10

Exercises 3-5a

(Class Discussion)

1. Trace the pair of data values



through the flow chart in Figure 10.

2. What is your instruction when you come to the newly added assignment box $\boxed{N \leftarrow N + 1}$?

3. Is it possible to follow this instruction?

4. What change in the flow chart must be made so that the instruction $\boxed{N \leftarrow N + 1}$ can be followed?

To solve this problem we give the variable "N" an "initial" or starting value. This must be done just once. Therefore, we put the instruction in an assignment box outside the loop, as shown in Figure 11a.

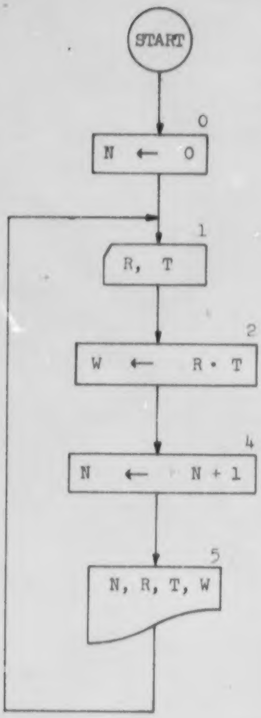

START
0
N ← 0
1
R, T
2
W ← R • T
4
N ← N + 1
5
N, R, T, W

Figure 11a

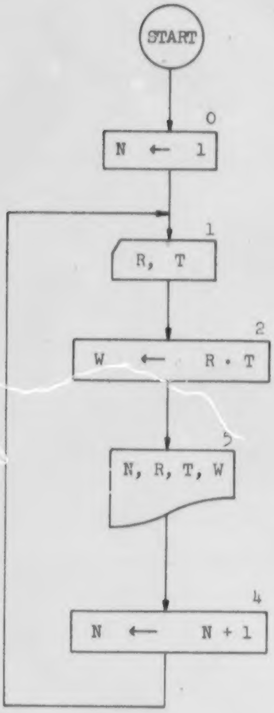

START
0
N ← 1
1
R, T
2
W ← R • T
5
N, R, T, W
4
N ← N + 1

Figure 11b

In Figure 11a, we assigned the variable N a starting value of 0 instead of 1. If we begin with the value 1, it would be stepped up to 2 in box 4 before any output. The first line of output would then be numbered 2, a result which of course we do not want.

It is possible to start N off with the value 1 if we rearrange the boxes. The flow chart in Figure 11b achieves the same result as that in Figure 11a. In Figure 11b we step up the value of N after, rather than before, the output step. This probably seems more natural. However, Figure 11a has the advantage that we can simplify it in the following manner:

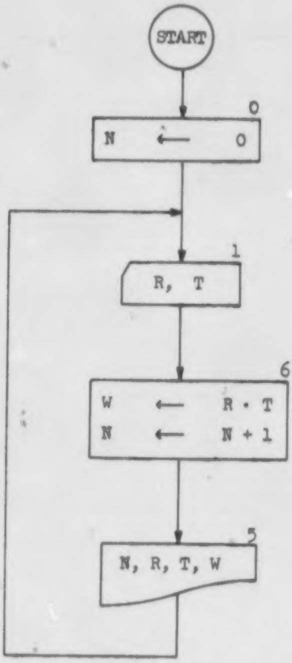


Figure 12

Whenever two or more assignments are called for in succession with no other steps in between, we will allow all the assignment steps to be put into one flow chart box with the understanding that these steps are to be carried out in order reading from top to bottom.

Exercises 3-5b

(Class Discussion)

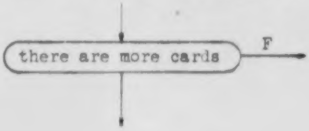
1. An employer using the flow chart of Figure 12 would also like to know the total amount of his payroll (that is, the total of all the wages paid). This can be accomplished by introducing a variable P (for payroll) into our flow chart. Each time a worker's wages are computed, the value of P is increased by the value of W.

   (a) Write the assignment statement that orders the Master Computer to increase the value of P by the value of W.

   (b) Write the assignment statement that assigns the starting value of P.

   (c) When will P have the desired value (that is, the sum of all the wages paid)?

   (d) Revise the flow chart of Figure 12 to include the above features and to provide for the output of only the final value of P. Hint: you will want to use the flow chart box:



Exercises 3-5c

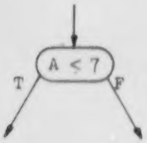
1. Using the flow chart developed in Class Discussion Exercises 3-5b, write the complete output for each of the following pairs of input data:

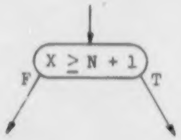
	R	T
(a)	2.50	32
(b)	3.00	38
(c)	3.40	22
(d)	2.75	40
(e)	3.60	39

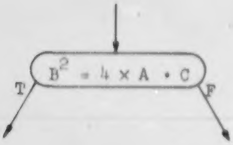
3-6. Decision and Branching

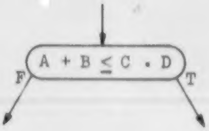
In our informal "flat tire" and "lost in the woods" flow charts we have seen decision boxes. Each decision box with its two exits introduces a branching. Often one or both branches lead into a loop. We have already noted the importance of looping for computers.

We are ready to consider mathematical flow charts involving decision boxes. The statements that appear inside these decision boxes are mathematical statements, either equations or inequalities. Some examples are:

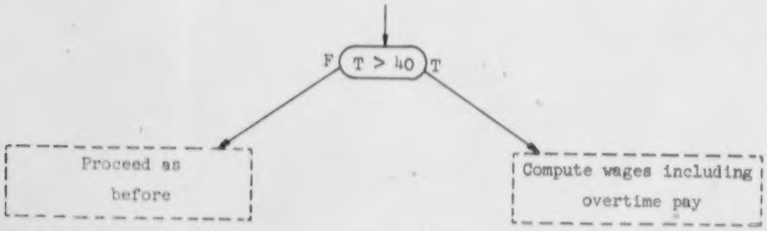








As a simple example of the use of a decision box, suppose that we wish to include an overtime feature in our payroll flow chart. If an employee receives double pay for all hours worked over forty, then we need the following decision box in our flow chart:


T > 40
F
T
Proceed as before
Compute wages including overtime pay

Before proceeding with the development of our flow chart we need to find a formula for the wages of an employee which will include the pay he receives for working overtime. The following exercises will develop such a formula starting with the familiar formula

$$W = R \cdot T$$

where W represents his wages, R his pay per hour, and T the number of hours worked.

Exercises 3-6a

(Class Discussion)

1. If the employee works 40 hours for R dollars an hour, then write an expression that represents his wages for the 40 hours.
2. The employee is paid double the hourly rate when he works overtime. Write an expression representing his rate of pay per hour for overtime where R is his regular hourly rate.
3. If T represents the total number of hours worked, then write an expression which represents the number of hours of overtime (i.e., the number of hours he works in excess of 40 hours).
4. Having found the rate of pay for overtime, (Exercise 2), and the number of hours of overtime, (Exercise 3), write an expression representing the wages for overtime work.
5. To the regular wages of the employee we must add his wages for overtime work. Write an expression which represents the total wages of the employee that includes the pay he receives for working overtime.

By using the assignment box

$$\boxed{W \leftarrow R \cdot 40 + 2 \cdot R \cdot (T - 40)}$$

we obtain the following flow chart which provides for extra pay for overtime, for output of the weekly wages of any employee, and for the total amount in the whole payroll.

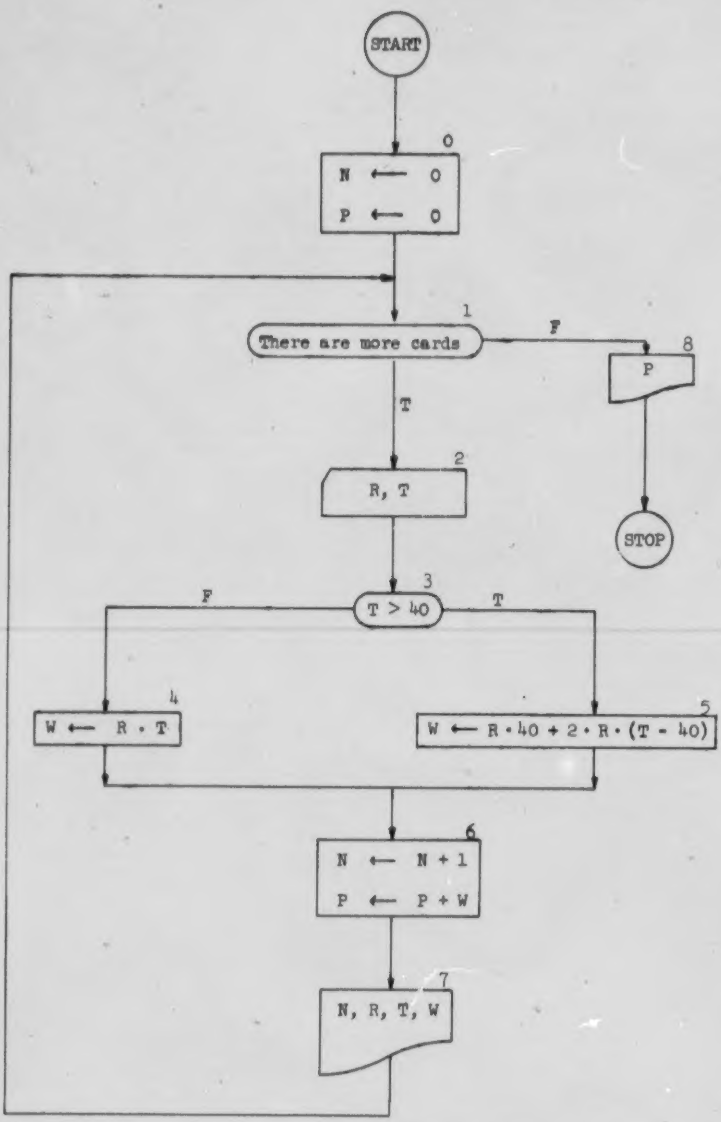

START
0
N ⟵ 0
P ⟵ 0
1
There are more cards
F
8
P
T
STOP
2
R, T
3
T > 40
F
T
4
W ⟵ R · T
5
W ⟵ R · 40 + 2 · R · (T - 40)
6
N ⟵ N + 1
P ⟵ P + W
7
N, R, T, W

Figure 13. Flow chart for payroll, including double rate for overtime.

Notice that there are two output boxes in Figure 13, one for the wages of the individual employee and one for the total payroll.

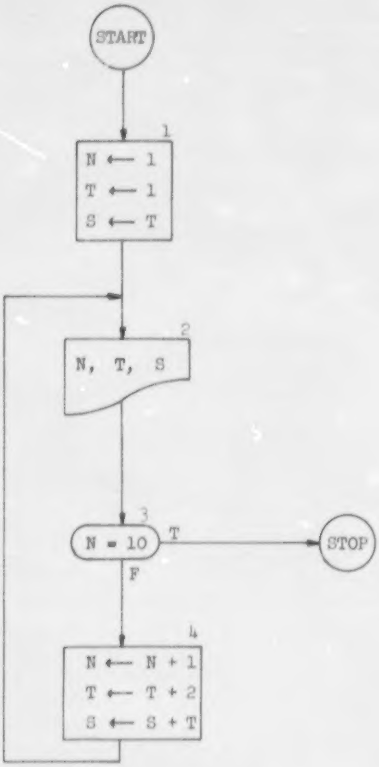
Exercises 3-6b

1. Trace through the flow chart of Figure 13 with the input given below. Give the output for each time through the output box 7, and finally the output for the total payroll (P), box 8.

	R	T
First card	2.15	39
Second card	2.64	44
Third card	1.98	27
Fourth card	2.15	40
Fifth card	2.26	45

2. If two assignment statements occur in the same assignment box, give conditions under which the two statements may be interchanged without changing the values which will be assigned to any variables.

3. Trace through the following flow chart and give the output values. In carrying out such a trace you should have a piece of paper on which to list the output values and a scratch pad on which you keep a running record of the latest value assigned to each variable. Each time you assign a new value to a variable, cross out the old value and write down the new one. The appearance of the output sheet and the scratch pad are given below for the first three times through the loop.



OUTPUT

N	T	S
1	1	1
2	3	4
3	5	9
4	7	16

SCRATCH PAD

N	~~1~~	~~2~~	~~3~~	4
T	~~1~~	~~3~~	~~5~~	7
S	~~1~~	~~4~~	~~9~~	16

4. Answer these questions about Exercise 3.

   (a) Describe in words the output list of values of N.

   (b) Do the same for the list of output values of T.

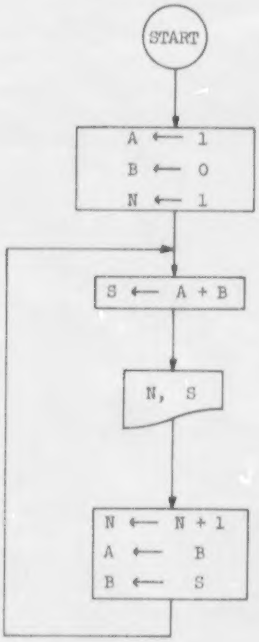
   (c) Each value in the list of values of S (after the first one) can be found by adding what other two numbers in the output list?

   (d) What instruction in the flow chart illustrates your answer to part (c)?

   (e) Can you express the output values of S entirely in terms of the various output values of T?

   (f) Fill in the blanks. The result of part (e) can be expressed by saying that the purpose of the variable S is to keep a running __________ of the values of __________.

5. Trace through the accompanying flow chart and give the output. Carry your work to the stage where N has the value 15.



6. Recall that 2^5 represents $2 \times 2 \times 2 \times 2 \times 2$ and is equal to 32. Similarly, 2^K, K a counting number, represents the product

$$2 \times 2 \times 2 \times \ldots \times 2$$

where there are K factors all equal to 2. This number is called the Kth power of 2. We want to make a flow chart to output each power of two from the first through the 20th. We cannot use an instruction with a string of dots in it and we will not permit the use of exponents. The table below will help in figuring out some correct instructions. (P represents the value of the Kth power of 2.)

K	1	2	3	4	5	6			
P	2	4	8	16	32	64			

(a) How is each value of K obtained from the preceding one?

(b) How is each value of P obtained from the preceding one?

(c) Fill in at least five more columns in the table.

(d) Now make your flow chart. Be sure to give K and P starting values and also provide for a stopping mechanism.

7. This problem is similar to problem 6. The number "five factorial" is written 5!, and means $1 \times 2 \times 3 \times 4 \times 5$ which is equal to 120. Similarly, if K is a counting number, then K! is defined as

$$1 \times 2 \times 3 \times 4 \times \ldots \times K,$$

that is, the product of all counting numbers from 1 through K. As in problem 6 we tabulate a few values here. (F represents the value of K!)

K	1	2	3	4	5		
F	1	2	6	24	120		

(a) How is each value of K obtained from the preceding value?

(b) How is each value of F obtained from the preceding value of F and the current value of K?

(c) Fill in two more columns in the table.

(d) Draw your flow chart. Arrange to stop when 15 values are printed out.

Exercises 3-6c

(Class Discussion)

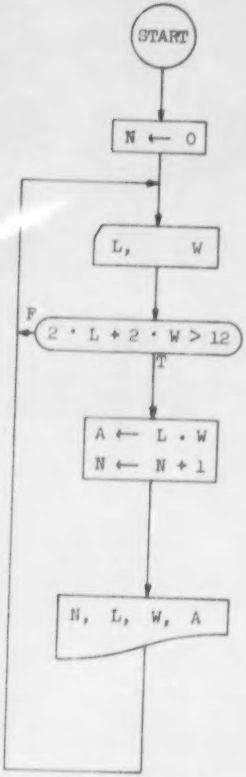
A teacher assigned her students a problem of constructing a flow chart as follows:

(1) The input consists of the lengths and widths of several rectangles.

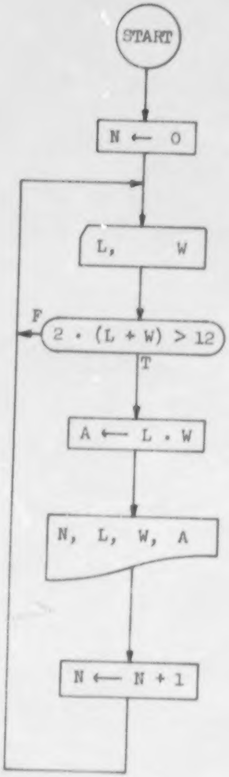
(2) The purpose is to output a list of consecutively numbered lines, starting at one, giving the length, the width, and the area of only those rectangles with perimeter greater than 12.

The flow charts shown on the next two pages were submitted by students as solutions of the problem.

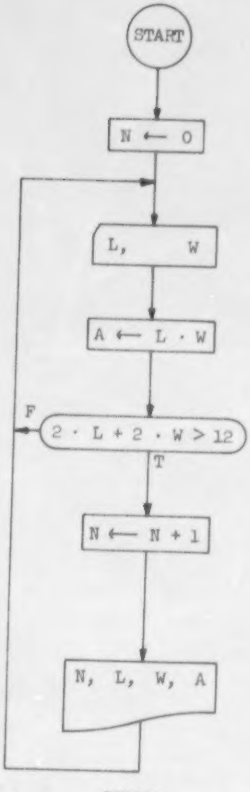
1. Which of the solutions are correct, and which are incorrect?
2. For those that are incorrect, in what way will the answers produced be wrong?
3. For those which are correct, list them in order of efficiency with the one requiring the least amount of computation first.


START
N ← 0
L, W
F
2 · L + 2 · W > 12
T
A ← L · W
N ← N + 1
N, L, W, A

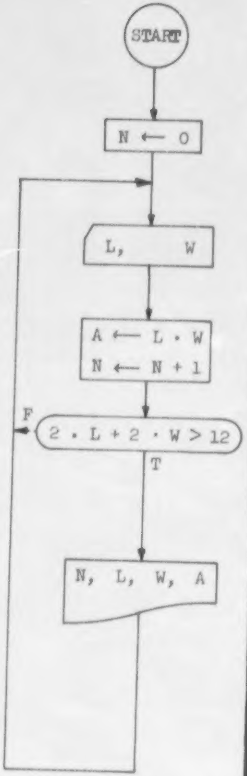
JOHN


START
N ← 0
L, W
F
2 · (L + W) > 12
T
A ← L · W
N, L, W, A
N ← N + 1

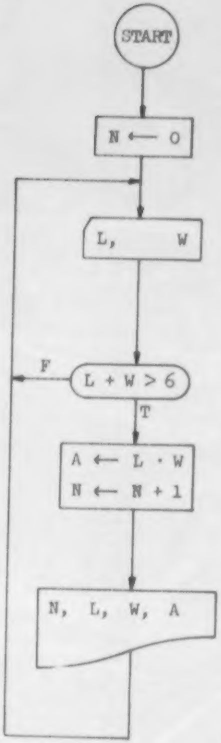
PAUL


START
N ← 0
L, W
A ← L · W
F
2 · L + 2 · W > 12
T
N ← N + 1
N, L, W, A

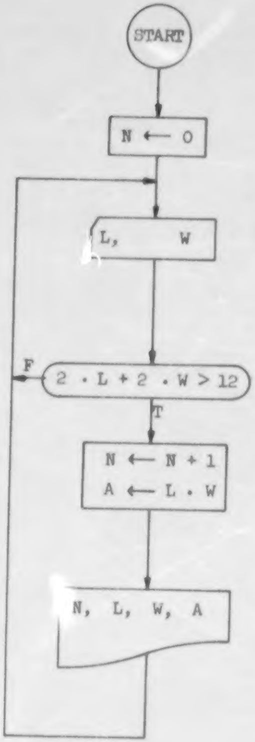
GEORGE


START
N ← 0
L, W
A ← L · W
N ← N + 1
F
2 · L + 2 · W > 12
T
N, L, W, A

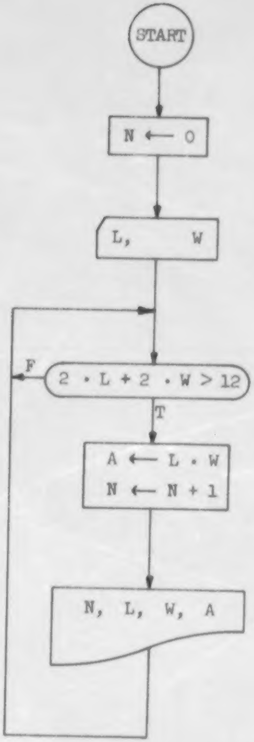
PETE


START
N ← 0
L, W
L + W > 6
F
T
A ← L · W
N ← N + 1
N, L, W, A

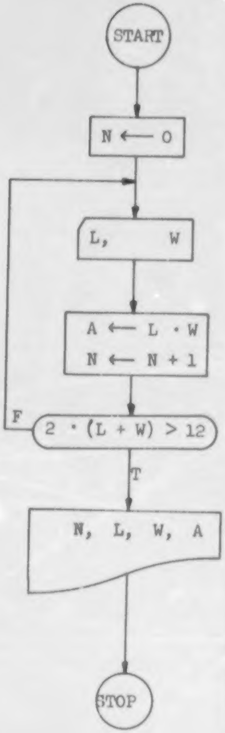
TOM


START
N ← 0
L, W
2 · L + 2 · W > 12
F
T
N ← N + 1
A ← L · W
N, L, W, A

GORDY


START
N ← 0
L, W
2 · L + 2 · W > 12
F
T
A ← L · W
N ← N + 1
N, L, W, A

LARS


START
N ← 0
L, W
A ← L · W
N ← N + 1
2 · (L + W) > 12
F
T
N, L, W, A
STOP

BOB

In the previous exercises you have seen that there are several correct flow charts for the desired algorithm. Usually we want to find the most efficient one from a computational viewpoint, but sometimes we want another feature included which may not lead to the least number of calculations. The basic requirement for any flow chart is still, "Will it work?" Simplifying and "streamlining" the flow chart can be accomplished as needed.

3-7. Flow Charting the Division Algorithm

A playground director found a sack of marbles while cleaning up the storeroom. Instead of throwing them away, he decided to divide them among the seven boys on the playground who were helping him. If, after dividing the marbles equally among the boys, there were any left over, he would put the extras away for the time being.

Here is the way the director distributed the marbles. First he lined the boys up.

John
Paul
George
Pete
Tom
Gordy
Lars

Then he reached into the sack and took out seven marbles and put one marble in front of each boy. He repeated this process over and over.

Let us take a look at this process somewhere in the middle. We see that the marbles distributed form a rectangular array.

John ○ ○ ○ ○ ○ ○
Paul ○ ○ ○ ○ ○ ○
George ○ ○ ○ ○ ○ ○
Pete ○ ○ ○ ○ ○ ○
Tom ○ ○ ○ ○ ○ ○
Gordy ○ ○ ○ ○ ○ ○
Lars ○ ○ ○ ○ ○ ○

We know that the total number of marbles in the array is equal to the number of rows times the number of columns (rows are horizontal; columns are vertical). As there are seven boys there are seven rows. Let Q be a variable representing the number of columns. Then the number of marbles distributed so far is

$$7 \cdot Q.$$

We see that the number of marbles already distributed plus those remaining in the sack is equal to the total number of marbles that the director found in the storeroom. Thus, if we let R be a variable representing the number of marbles remaining in the sack and let M represent the number of marbles he had at the beginning, we have the formula:

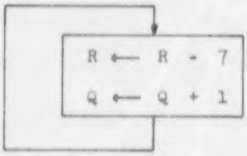
$$\underbrace{M}_{\substack{\text{Total}\\ \text{number of}\\ \text{marbles}}} = \underbrace{7 \cdot Q}_{\substack{\text{number}\\ \text{distributed}}} + \underbrace{R}_{\substack{\text{number}\\ \text{remaining}\\ \text{in sack}}}$$

This formula is true at every stage of the distribution process.

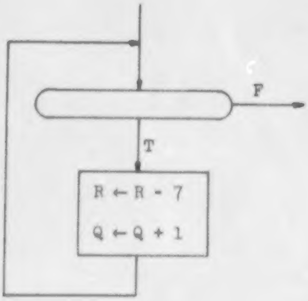
We can recognize the director's process as an algorithm and we will draw a flow chart for it. The basic step is taking seven marbles out of the sack and using them to form a new column of the array. In doing this we decrease R (the number in the sack) by 7 and increase Q (the number of columns) by 1. These activities are represented by the assignment statements:

$$\boxed{\begin{array}{l} R \longleftarrow R - 7 \\ Q \longleftarrow Q + 1 \end{array}}$$

Since this process is to be repeated over and over we write:



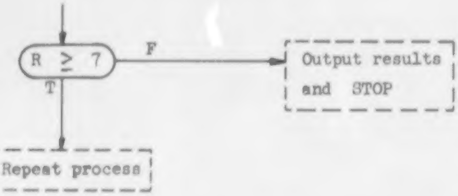
The difficulty with such a loop is that no way is provided for stopping. As you remember, in order to escape from this endless loop a decision box must be introduced.



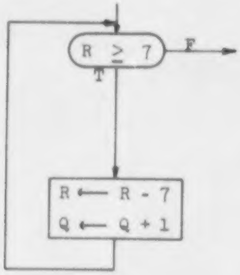
The statement to be placed in the decision box will stem from the fact that in distributing the marbles there comes a time when we cannot remove seven marbles from the sack because there will not be seven marbles left. In other words, in order to remove seven marbles it is necessary that the remainder (of marbles) be greater than or equal to seven, that is,

$$R \geq 7.$$

If we place this statement in the decision box we will provide a means for stopping the process.

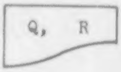


Combining this with the above assignment box we have:

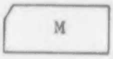


That is the heart of our flow chart. Only minor details remain: namely, to provide output and to give starting values to our variables.

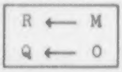
As output we merely give the values of Q and R, thus:



We want to input the value of M:



We start R and Q out with the values they should have before any marbles have been distributed. These are given by:



Putting all our flow chart fragments together we get the complete flow chart.

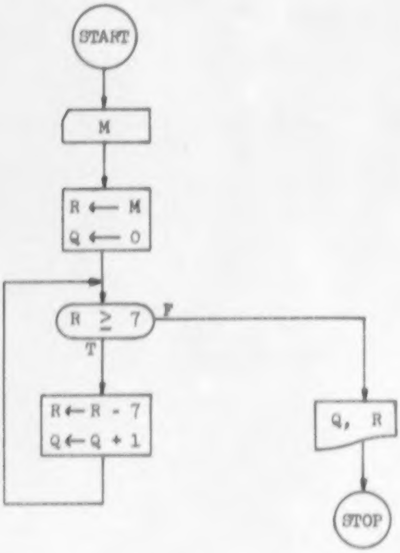


Figure 14

The final value of Q, the value that is output, is the number of columns in our final array of marbles--the number of marbles each boy gets. The final value of R is the number of extra marbles kept by the director and is one of the numbers

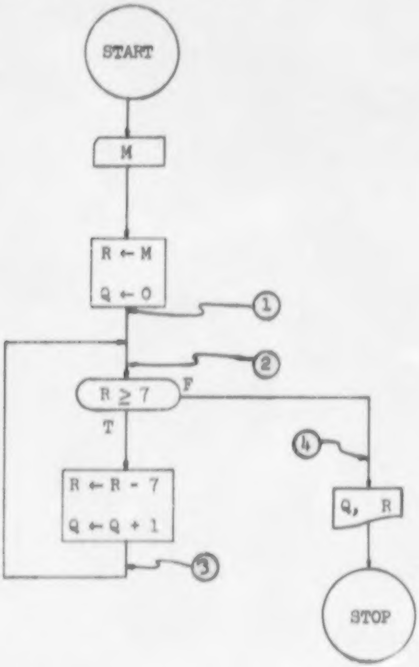
0, 1, 2, 3, 4, 5, 6.

The algorithm can be called the algorithm for integer division with a divisor of 7. We have divided M marbles into 7 piles of Q marbles each with R marbles left over. This is the best distribution that can be made without breaking any marbles.

Exercises 3-7a

(Class Discussion)

1. The flow chart of Figure 14 is shown below.



Using an input value of 17, find the values of the variables Q and R at each of the following stages:

(a) The first time we arrive at the point marked ①.

(b) The second time we arrive at ②.

(c) The second time we arrive at ③.

(d) The third time we arrive at ②.

(e) The first time we arrive at ④.

We see that there are four numbers involved in this integer division process.

$$\underbrace{M} = \underbrace{7} \cdot \underbrace{Q} + \underbrace{R}$$

Dividend:	Divisor:	Quotient:	Remainder:
number of things to be divided.	number of piles into which the dividend is to be divided.	number of things in each pile.	number of things remaining undistributed (remainder is less than divisor).

Of course, the same kind of reasoning would work f[illegible]ny divisor. The divisor does not have to be 7. To make a flow chart for [illegible]eger division for any divisor, we use a variable, D, to denote the divisor. We call for both the dividend and the divisor to be input, and we replace each 7 in the preceding flow chart by D. Then we will have the flow chart:

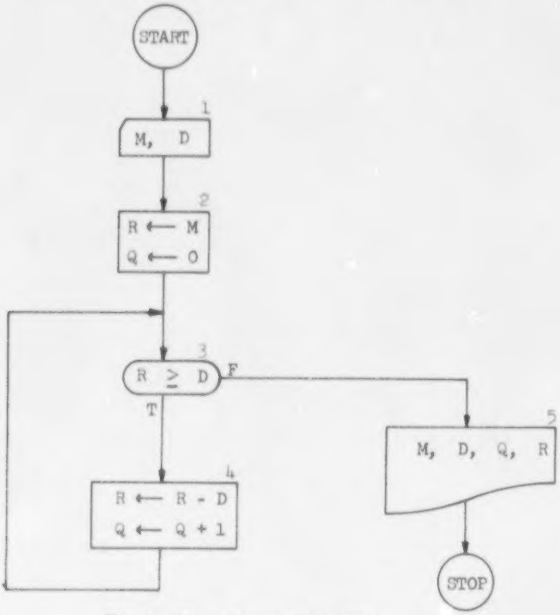
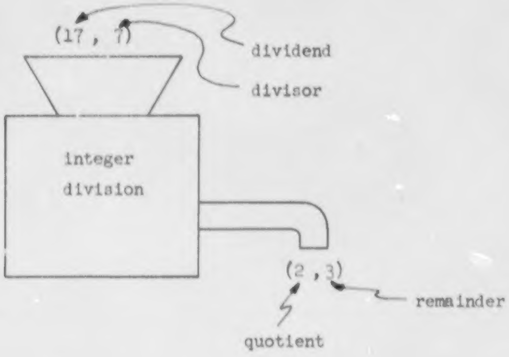


Figure 15. Integer division

This time we have called for the output of the values of all the variables M, D, Q, and R in this order, so as to suggest the formula

$$\underbrace{M}_{\text{Dividend}} = \underbrace{D}_{\text{Divisor}} \cdot \underbrace{Q}_{\text{Quotient}} + \underbrace{R}_{\text{Remainder}}.$$

We can think of integer division as a function whose input is a pair of numbers (the dividend and the divisor) and whose output is another pair of numbers (the quotient and the remainder).



The dividend and the divisor are a pair of whole numbers (with the divisor not equal to 0). The quotient and the remainder form another pair of whole numbers--the only pair of whole numbers Q and R satisfying the two conditions

(a) $M = D \cdot Q + R,$

(b) $R < D.$

During an application of the algorithm the first of these conditions holds at every stage. We repeatedly decrease the value of R by subtracting D until the second condition is satisfied.

Exercises 3-7b

1. Tell what will happen in our algorithm for integer division if we disobey instructions and input the value 0 for D.

2. Trace through the flow chart of Figure 15 with each of the following pairs of input values. For each pair of input values make a table showing the values of all four variables at each passage through box 3 of the flow chart. The last line in the table will be the output of the flow chart. In each of these pairs the dividend is given first and the divisor second.

   (a) (23 , 7)
   (b) (24 , 5)
   (c) (5 , 24)
   (d) (0 , 23)
   (e) (24 , 8)
   (f) (73 , 6)

   We give the solution to part (a) as an example:

	M	D	Q	R
1st	23	7	0	23
2nd	23	7	1	16
3rd	23	7	2	9
4th	23	7	3	2

3. In each of the parts of the preceding problem you note that the values of M and D never change. What is it about the flow chart which explains this?

4. Given the following pairs of values for M and D perform the necessary division and find the corresponding pairs of output values for Q and R.

   (a) (17 , 7)
   (b) (39 , 13)
   (c) (49 , 2)
   (d) (54 , 9)
   (e) (53 , 53)
   (f) (17 , 35)
   (g) (38 , 12)
   (h) (125 , 125)
   (i) (194 , 64)
   (j) (84 , 12)
   (k) (14 , 1)
   (ℓ) (62 , 1)
   (m) (272 , 16)
   (n) (3168 , 7)
   (o) (958 , 236)

5. (a) What will be the output values of Q and R if M and D have the same value?

(b) What will be the output values of Q and R if the input value of D is 1?

(c) If you are given the output values of Q and R, are you able to determine what the input values of M and D were? Explain.

3-8. Summary

Section 3-1.

An *algorithm* is a list of instructions for carrying out some process in a step-by-step, sequential manner.

A *flow chart* is a diagram which represents the steps in an algorithm.

In a flow chart, *commands to take some action* are enclosed in *rectangular* boxes.

In a flow chart, *statements* on which we are asked to make some *decision* are enclosed in *oval* frames, and these boxes always have two exits.

In a flow chart, a *loop* is a convenient method of handling a repetitive process, but there must be some way out of the loop to prevent it from becoming an endless process.

Section 3-2.

When preparing to solve a problem with a computer it is usually necessary to construct an algorithm for the problem. In preparing the algorithm it is not necessary to know exactly what the computer will do at each step, but it is necessary to provide instructions which, if followed, will lead to the correct answer. In other words, in constructing a flow chart we try to provide for all of the alternative paths that the computer might take even though we do not know exactly what paths our computer will follow at any given stage.

Section 3-3.

In any computing problem, there corresponds to each *variable* used in that problem a location in the computer's memory. By *assigning* a number to a variable we mean simply putting the number destructively into the storage location corresponding to that variable. In evaluating arithmetic expressions a variable is to be treated as a name for the number in the corresponding memory location. The number in the corresponding memory location is referred to as the value (or current value) of the variable. During the course of computation many different values (perhaps even millions) may be assigned to a given variable. Thus it will not be meaningful to speak of the value of a variable without specifying the time, or more precisely, the stage of the computing process. But once the stage of the process is specified, the value of the variable is uniquely determined.

An assignment statement is always placed in a rectangular box like the following: $\boxed{I \leftarrow P \cdot R \cdot T}$. We read this statement

"assign to I the value of $P \cdot R \cdot T$".

This flow chart box is called an assignment box.

Assignment is destructive in that it destroys the former value of the variable. Reading the value of the variable is not destructive.

Section 3-4.

An output box, in a flow chart, contains a single variable or a list of variables which is a command to read the values of the variables and print out these values in the order listed.

An input box, in a flow chart, contains a single variable or a list of variables and is a command to assign the value of the variables, in the given order, to the appropriate place in the memory. Another function of the input box is to stop the computer when all of the data cards have been used.

Section 3-5.

Sometimes we wish to count the number of sets of data assigned to the memory or number the acceptable outputs. We use a variable in an assignment box like the following to accomplish this process.

$$\boxed{N \leftarrow N + 1}$$

You must be sure to assign the variable an initial value such as $\boxed{N \leftarrow 1}$ or $\boxed{N \leftarrow 0}$ so that the counting can begin.

Section 3-6.

Branching is indicated in flow charts by a decision box that is oval in shape. The decision box gives us the ability to choose a new path depending on whether a certain condition is satisfied. One of the functions of the decision box is that it enables us to get out of an endless loop in a flow chart.

Section 3-7.

In flow charting the division algorithm for integer division we think of it as a function whose input is a pair of numbers (the dividend and the divisor) and whose output is another pair of numbers (the quotient and the remainder).

When we input a value of 0 for D in our flow chart (that is, try to divide by zero) we get locked into an endless loop in which the quotient increases indefinitely.

Chapter 3

INFORMAL ALGORITHMS AND FLOW CHARTS

General Comments

A method of communication between an individual and the computer has been accomplished by the development of procedural languages. They have as their purpose the task of relating to the computer some "unambiguous plan telling how to carry out a process in a finite number of steps." The algorithm (as the plan is called) requires a highly sequential step-by-step method of computing an answer or answers to a problem. An early link in this line of communication is the flow chart.

Simply stated:

(1) An algorithm is a procedure to solve some problem or problems.

(2) A flow chart is one expression of an algorithm.

One should bear in mind that the flow chart, though less formal than the machine language, requires the student to put down a series of low order steps (with well disciplined care and forethought) for a simple minded machine.

An expression of how to do something is essentially what is meant by a flow chart; however, it is expedient for pedagogic reasons, as well as our desire to have this work computer oriented, to adopt suitable flow chart language.

Language is a means of expressing and communicating our ideas. In computing we want to be able to communicate not only with people but also with computers. To communicate with people we normally use the "natural language" we have learned as children. Still, in specialized topics, people have always found it useful to devise specialized jargons and languages. You will discover that the flow chart language helps us to develop, display, and discuss algorithms in an unambiguous way.

We know of no other way to express algorithms that can be as effective as a flow chart in highlighting significant points. After an algorithm has been expressed as a flow chart, an equivalent machine language program must be developed so that a computer can execute the algorithm. However, the translation from flow chart language to a machine language becomes nearly painless by exploiting a procedural language as an intermediate stage. Conversion of an algorithm from flow chart language to a procedural language is, for the most part, a simple translation process. Translation from a procedural language to a machine language is done automatically by a computer program called a compiler.

Learning and using the flow chart language will encourage the development and use of a problem solving technique of tremendous power. This process will demonstrate how to subdivide relatively complicated problems into simpler, more easily understood subproblems. This is a technique vital to solving problems in all walks of life.

NOTE TO TEACHERS

Based on previous experience and the estimates of the authors of the preliminary edition (including junior high school teachers), it is recommended that the teaching time for this chapter be no more than 10 days, including testing.

Teachers are reminded that complete mastery of the concepts and processes in this chapter should not be expected. Reinforcement and further development of these concepts and processes will occur in a spiral manner in future chapters.

Teachers are urged to try not to exceed the suggested time allotment so that pupils will not miss the chapters at the end of the course.

3-1. Changing a Flat Tire

The idea of an algorithm and a flow chart has been purposely introduced in a non-mathematical setting. Hopefully, the student will learn that most processes can be broken down into a series of directed steps and also that the student will learn to follow these steps. Also, it should be clear that there can be a great variety of flow charts all of which can result in a correct solution.* This concept could be the first step in helping the

---

*Note: In industry, many programs may be submitted for the solution of a single problem. It is the function of the master programmer to determine which, of those submitted, will work the most efficiently and thus the most inexpensively.

the student avoid rigid thinking in mathematics.

The language and notation used in this section is *not* the final flow chart language or notation. Therefore do not emphasize the language or notation at this time. Emphasis, if any, should be placed on the step-by-step, sequential development of the flow chart illustrated in the Class Discussion, Exercises 3-1.

The *decision box* and the *loop* are two bits of flow chart notation that are in final form. At this stage of the development though, the language which they contain is not. Students should know that the *decision box* should contain a *statement* (not a question) that can be answered either T (true) or F (false) and that there *must be two exits* from this box. The basic idea of the *loop* is that it provides a simple means to perform repetitive steps. The function of the loop should be emphasized to the students.

Finally, students should be made to realize that if the flow chart of their algorithm does not provide a solution *every time*, then *they have not written a real algorithm*.

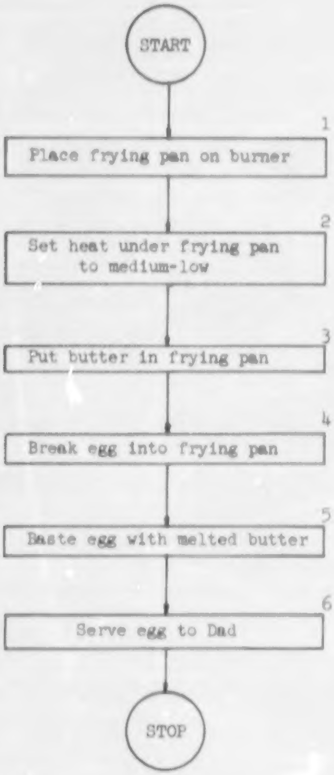
Exercises 3-1 (Class Discussion)

It is suggested that the instructor develop this exercise either on the chalkboard or with the overhead projector. Students should develop their flow chart fragments first, before the instructor provides a suitable model of the fragment.

Teaching suggestion:

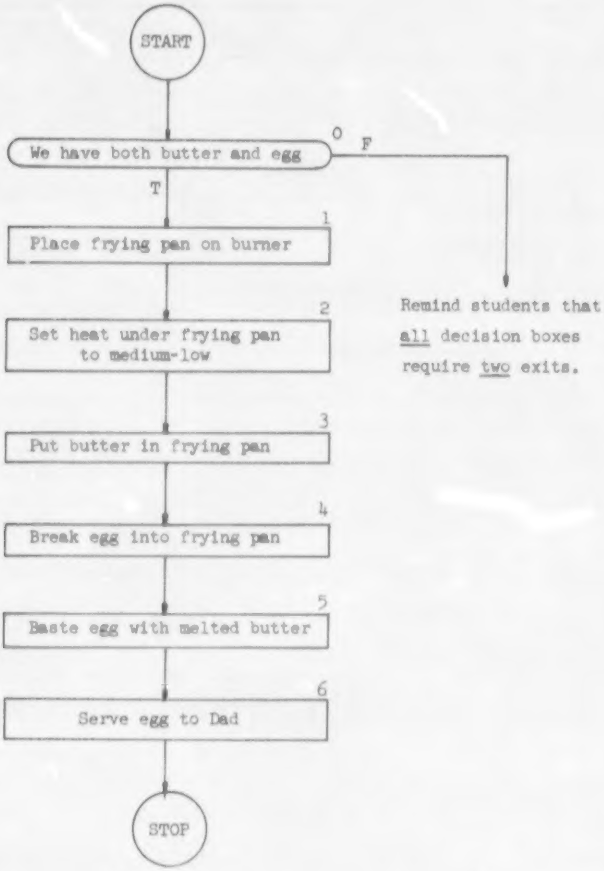
If you develop these flow charts on the chalkboard be sure that the initial flow chart has ample room between boxes to insert the required refinements as you progress through the development.

1. (a)


START
1
Place frying pan on burner
2
Set heat under frying pan to medium-low
3
Put butter in frying pan
4
Break egg into frying pan
5
Baste egg with melted butter
6
Serve egg to Dad
STOP

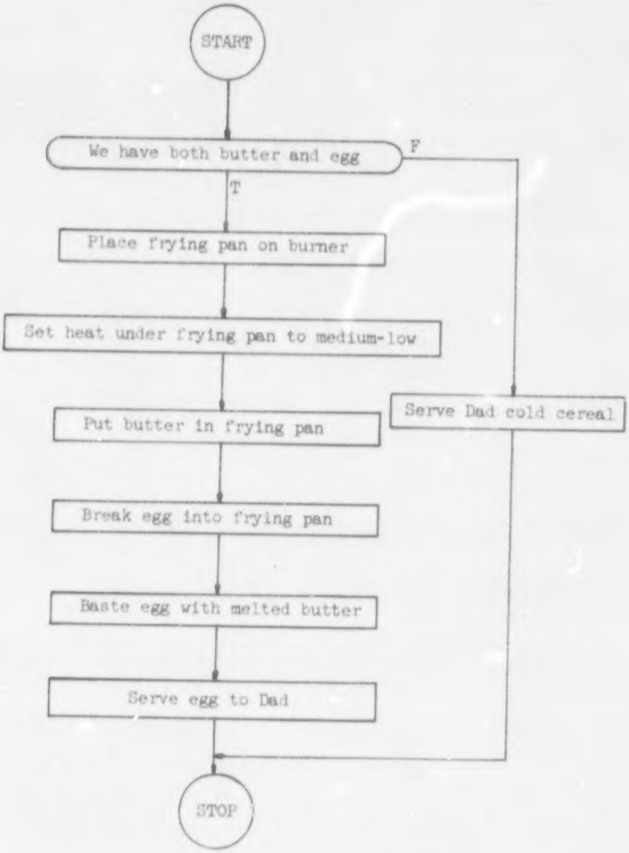
Students may have drawn the flow chart but omitted the START and STOP. This will provide the opportunity to remind the students that the machine must be started and stopped in some manner.

(b)


START
0
We have both butter and egg
F
T
1
Place frying pan on burner
2
Set heat under frying pan to medium-low
3
Put butter in frying pan
4
Break egg into frying pan
5
Baste egg with melted butter
6
Serve egg to Dad
STOP

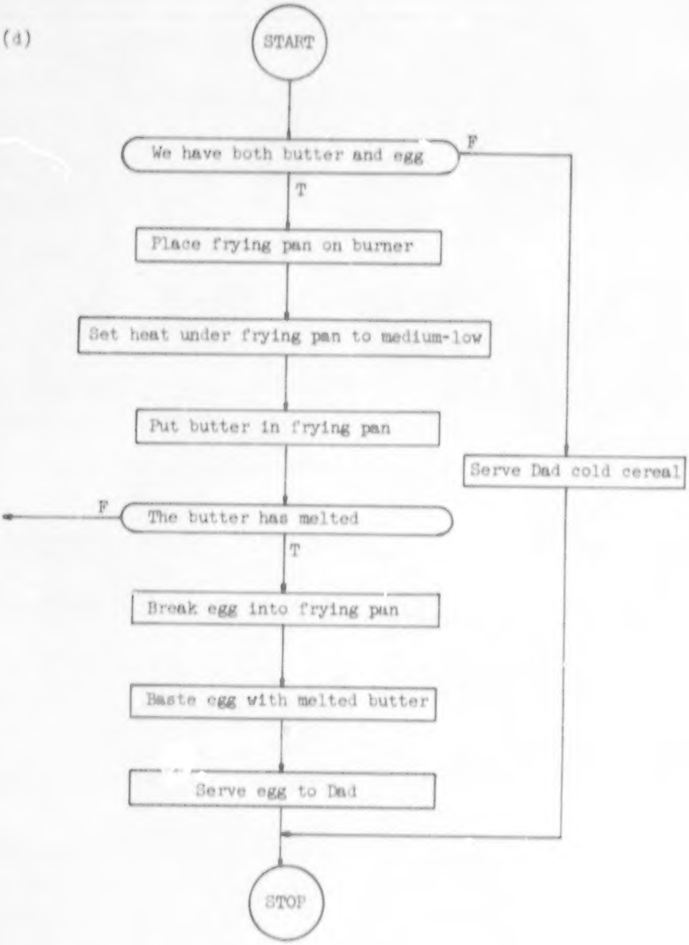
Remind students that
__all__ decision boxes
require __two__ exits.

(c)


START
We have both butter and egg
F
T
Place frying pan on burner
Set heat under frying pan to medium-low
Put butter in frying pan
Break egg into frying pan
Baste egg with melted butter
Serve egg to Dad
Serve Dad cold cereal
STOP

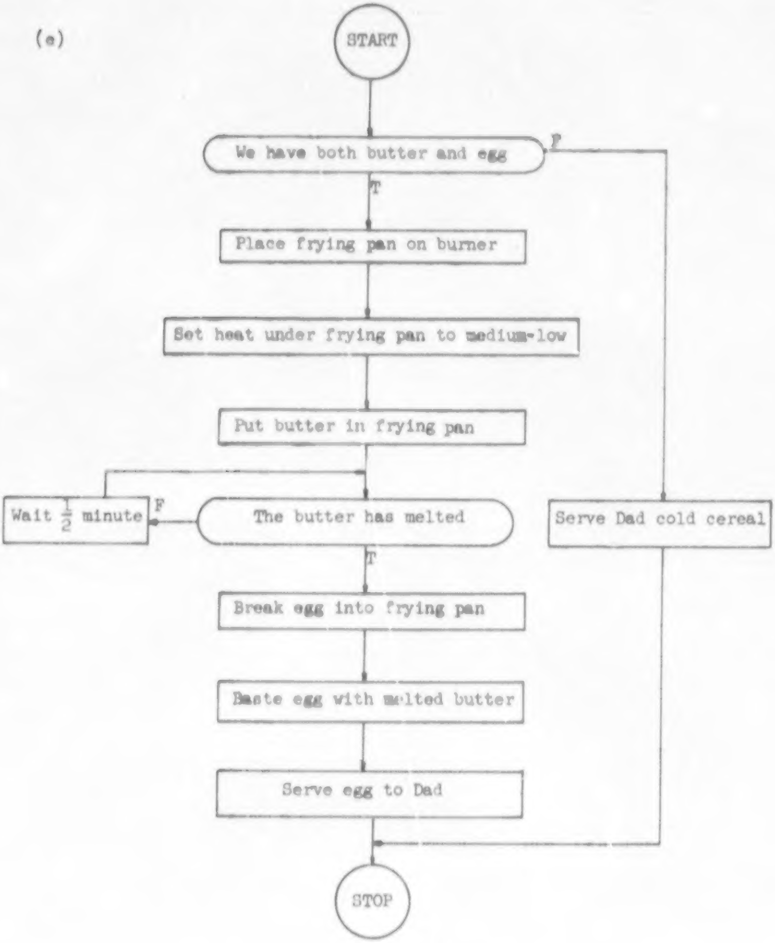
Here it would be well to reiterate that not only does a decision box require two exits but that both exits must lead to some reasonable portion of the chart.

(d)


START
We have both butter and egg
F
T
Place frying pan on burner
Set heat under frying pan to medium-low
Put butter in frying pan
Serve Dad cold cereal
F
The butter has melted
T
Break egg into frying pan
Baste egg with melted butter
Serve egg to Dad
STOP

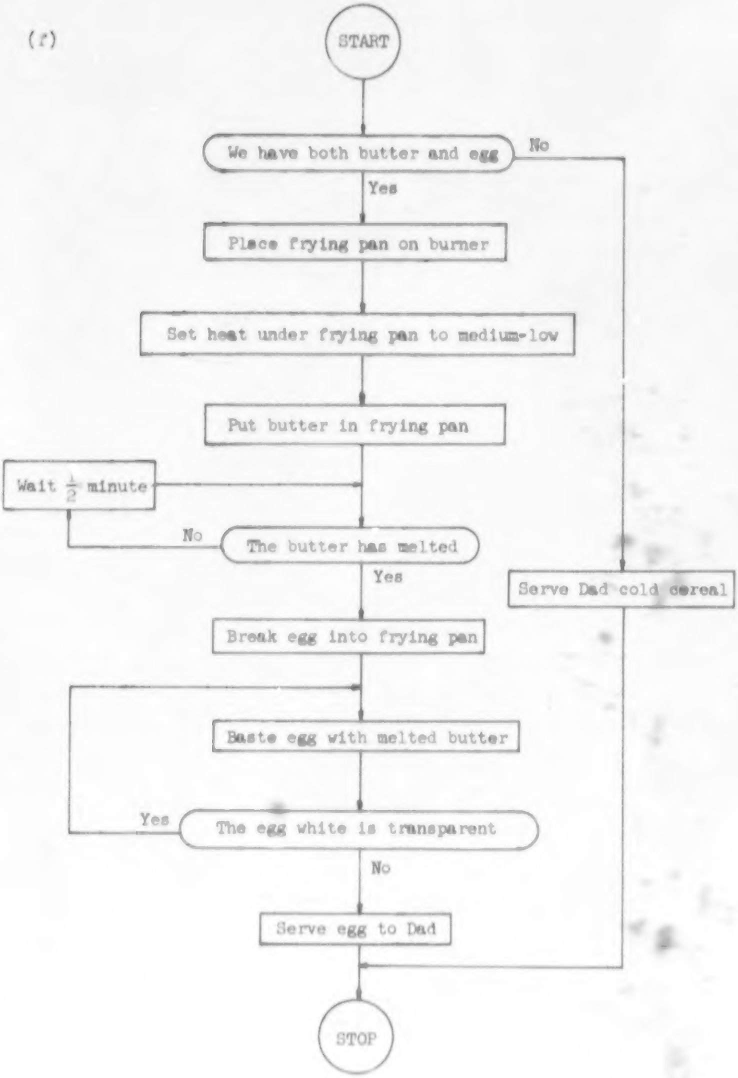
It might be well to spend some time in discussion of just where the F (false) exit will lead and what to do if "the butter *has* [*not*] melted". For example, with heat under the pan and butter in the pan there is no reason to assume it won't melt *in time*. See if the students can be led to suggest that waiting will result in the butter melting.

(e)


START
We have both butter and egg
F
T
Place frying pan on burner
Set heat under frying pan to medium-low
Put butter in frying pan
Wait $\frac{1}{2}$ minute
F
The butter has melted
T
Serve Dad cold cereal
Break egg into frying pan
Baste egg with melted butter
Serve egg to Dad
STOP

The time, $\frac{1}{2}$ minute, is purely arbitrary. Remind students that waiting too long, though, may result in burnt butter.

(f)


START
We have both butter and egg
No
Yes
Place frying pan on burner
Set heat under frying pan to medium-low
Put butter in frying pan
Wait $\frac{1}{2}$ minute
No
The butter has melted
Yes
Serve Dad cold cereal
Break egg into frying pan
Baste egg with melted butter
Yes
The egg white is transparent
No
Serve egg to Dad
STOP

At this stage of the development students should have little difficulty inserting the decision box in the required position and looping back correctly.

3-2. Algorithms, Flow Charts, and Computers

In this section we try to motivate the central topic of this chapter, algorithms. We are interested in details of actual computers only to the extent that they are necessary to motivate or explain certain aspects of the study of algorithm construction. Some students will want to know much more about how computers are built or how they work. Questions on the construction of computers and how they function are considered to be out of the scope of this chapter and particularly this section. However, some material and some references pertinent to these topics will be included in subsequent portions of the chapter, and these may help you satisfy the questions of your students.

A common characteristic of algorithms is that the step-by-step plan must be unambiguous. We cannot tell a computer to "either add or subtract." Rather, we must say -- "if specific conditions are satisfied, then add; if these specific conditions are not met, then subtract." There can be no room for doubt as to the meaning of the steps in the algorithm.

A second common characteristic of useful algorithms is repetition. In a musical composition, special symbols are used to tell a performer to repeat a part of the piece. In the same way, it is extremely useful to repeat a series of steps in a computing algorithm. One reason computers are so useful is that algorithms do depend heavily on repetition and the computer will repeat the same steps tirelessly and without complaint.

Exercises 3-2a (Class Discussion)

There are really two understandings that can be developed in these exercises. They are:

(1) An algorithm is a step-by-step set of instructions which guarantees a successful solution to the problem, but it is not necessary to know explicitly what the computer will do at each step, and

(2) There are many possible "good" algorithms. Some of the algorithms may appear to be more efficient than others, but many times the choice as to which is "better" is an arbitrary judgment.

1. (a) Follow the river downstream.

(Of course other commands are possible, such as, "Follow the river upstream." The answer given is the one that is generally acceptable to woodsmen, but there could be occasions where other directions would be more appropriate. For example, if the river flowed into the desert and disappeared, then the above command might not be too useful.)

(b) Stop. (He's reached civilization.)

(c) Follow the river downstream.

(d) Follow the shoreline of the lake, keeping the water on your left.

(It would be just as correct to say, "Follow the shoreline of the lake, keeping the water on your right.")

(e) Follow the river downstream.

(f) Follow the shoreline of the sea, keeping the water on your left.

(This procedure was chosen because it corresponds to what happened when the hiker came to a lake. As you will see in the flow chart, this enables us to combine these two steps in one decision box.)

(g) (The comparison can be valuable if the student is led to understand that there can be many different, acceptable algorithms for a given situation. You might point out that we used "body of water" to refer to both the sea and the lake.)

Exercises 3-2b

1. (a) Hunter A reaches Seaside
Hunter B reaches Woodland
Hunter C reaches Rock Beach
Hunter D reaches Seaside
Hunter X reaches Interlachen
Hunter Y reaches Rock Beach

(b) Hunters traveled distance (shortest to longest) in given order.

D
B
X
Y
C
A

(c) Hunters C and Y reach Rock Beach

(d) Hunters A and C

2. The treasure hunters should dig at point F.

We include in this section of the Teacher's Commentary a little historical background about computers. This article may help you answer some questions that might arise in your classroom. It is an excerpt from Algorithms, Computation and Mathematics, Revised Edition, SMSG, 1966.

"Man's efforts to ease the task of calculation can be traced back thousands of years. It is a story of famous people, such as Pope Urban II who introduced Arabic calculations into tenth century monastaries. Napier (the inventor of logarithms), Leibnitz, Pascal, as well as unknown contributors who developed the abacus, invented Arabic numerals and positional notation, and devised the first notions of arithmetic. Regretfully, we must skip over these ancient contributions as well as the more recent contributions of people such as the inventor Charles Babbage (19th century), the English mathematician Alan Turing in the 1930's, the American Howard Aiken in the 1930's and 1940's, and many others.

"Beginning in about 1945, a remarkable series of inventions and advances in understanding has made computing the fascinating and important topic that it is today.

"Four recent developments stand out as being instrumental in determining what the computer is and how it is used today. Briefly these are:

1. replacement of mechanical devices by electronic ones,
2. adoption of the "stored program" concept,
3. extensive use of "solid state" electronics, and

4. introduction of "procedural languages" as a means of expressing problems.

The first and third of these developments will be mentioned only briefly. They are largely engineering developments and our interest will be mainly in the mathematical aspects of using computers. Nevertheless, these developments have been important in determining the characteristics of the computers we now have.

"1. Electronic devices replaced mechanical devices in 1946, when J. P. Eckert and J. W. Mauchley at the University of Pennsylvania constructed a machine called ENIAC (for Electronic Numerical Integrator and Calculator). Earlier calculating machines had depended on some kind of mechanical movement like rotating gears (or toothed wheels), or switching electromechanical relays to perform arithmetic and record results. ENIAC was the first machine to rely basically on electronic devices (in this case, some 18,000 radio or vacuum tubes). For the first time the speed of an addition or multiplication was limited by the speed with which an electronic circuit could react to a signal rather than by the speed with which a wheel could be turned (or some other mechanical device moved). Now a thousand steps (think of them as additions) could be done in a second, as compared to ten or twenty steps per second before, a really tremendous difference! Suddenly, the solution of problems that required enormous computation became practical. Scientists could think about calculations ten, twenty or a hundred times more extensive than they had been able to think about earlier. Continuing improvements in the speed of electronic devices and circuits have now produced machines that can perform roughly a million steps in a second, perhaps a thousand times faster than ENIAC!

"2. The second major development that we have singled out is an idea suggested by John von Neumann and his associates in 1945. Calculating machines to that time (and including ENIAC) could store data for any problem in its "memory" but special wiring was needed to control the steps to be taken to operate on the data. Each new problem meant different wiring. Von Neumann suggested that numerals, to be treated as instruction codes, could be stored electronically and so eliminate the special wires. A machine could be designed to interpret a stored numeral as a code for a particular instruction. Once the type of instruction was determined it would establish connections between electronic circuits equivalent to the special wires that had to be added to earlier machines. Since a series of instructions controlling the behavior of a machine is called a program, this idea has come to be called the "stored program" concept.

"For machines built under the von Neumann plan the time and effort required to change a machine from solving one problem to solving a second one was drastically reduced. A computing machine could now be thought of as general purpose. It could help in solving a wide variety of problems without having to make external physical changes from problem to problem.

"A more subtle result of the stored program concept, but an important one, is that a series of instructions (i.e., a program) stored in the machine could, when executed, make changes in another sequence of instructions. That is, a program could make changes in another program, or, in fact, in itself. In this way, a program can be designed to modify itself to do different calculations depending on what has happened before.

"These two developments provide the basis for the modern device with the long-winded name: stored program digital electronic computer. (From now on we will refer to this as simply a computer.)

"3. The third development is the widespread use of "solid state" electronics in computers. Solid state, as used here, refers not only to the transistors and semiconducting diodes with which you are probably familiar from their use in radios, but also to certain magnetic ceramics, magnetic coatings and inks, and extremely thin films produced to have specific electronic characteristics. In fact, much current research in advanced electronics is stimulated by the desire for ever better, ever faster computers.

"The application of this research to computer design and construction has resulted in both increased speed and simplified fabrication. The net effect is that the cost of computing has decreased drastically (although computers are still very expensive indeed). Moreover, modern electronics has made the computer far more reliable than previously.

"In the early 1950's, when computers were made largely of vacuum tubes, one could reasonably hope for a few hours of operation before a tube failed. Sometimes a tube would fail after only a few minutes. Although vacuum tubes were later improved considerably, the use of solid state components throughout a computer has drastically reduced electronic failure. Failures still happen, but most often they are connected with mechanical equipment. Basically this means that a research problem can now run for an hour on a modern computer without fear of equipment failure, while the same problem would have taken as long as half a year on ENIAC even if there were no equipment failures to introduce further delays.

"The high level of reliability of modern computers has far reaching implications for the use of computers. Now it is practical to use computers in situations in which continuous twenty-four hour operation is mandatory, or where a computer malfunction would be potentially disasterous. Such situations arise in the automatic control of oil refineries, the monitoring of blood pressure (or other vital functions) of a patient in an operation, the control of air traffic around busy airports, or computerized telephone switching systems. In situations which can tolerate no failures at all, the computer is supported by an identical twin computer. The twin is always ready to take over instantly when and if the rare machine failure does happen.

"4. The series of coded instructions or numbers that make up a computer program must be written out in advance. To operate correctly, each instruction must be properly coded and must relate to the other instructions of the program in the intended way. As the computer developed, the task of preparing programs became the bottleneck in solving problems. It was not unusual for a person to spend several days preparing the code which took the computer only minutes to interpret and carry out. Computers got faster but people couldn't speed up. Of course, once coded, a program could be used as often as necessary to solve a given type of problem. So, with repeated use, the investment of a person's time was worthwhile. Nevertheless, some way surely had to be found to either slow down computers (obviously a step in the wrong direction) or to speed up the process of preparing programs.

"The notation of mathematics has been developed over hundreds of years to enable people to quickly and concisely describe ways of solving problems. If the computer could be controlled through the use of mathematical notation, people would have an easier and far more powerful way of expressing problems for computer solution. This is essentially what has been accomplished by the development of procedural languages. The use of such languages has had a major positive effect on our ability to express problems for solution on a computer, and is the subject of most of the rest of this book."

If you wish to learn more about how a computer operates you can read Chapter 1 and Appendix A of the book referred to at the beginning of this article. A simple hypothetical computer, SAMOS, is discussed and the details of how a computer operates are presented in these sections.

3-3. Assignment and Variable

The most fundamental ideas in this section are those concerned with *variables* and *assignments*. The concept of variable is very explicit. We feel that this is one of the more important by-products of flow charting. We hope and expect that the student will carry this interpretation of variable with him to other chapters.

The merit of our attitude toward variables in this context is that it enables the student to manipulate with more complicated arithmetic expressions without continually asking what they mean. He knows that the expression has a value which can be determined by looking up the values assigned to the variables and making the indicated computations. This visual model for interpreting expressions relieves the student of the necessity of knowing at all times the value of the expression.

This attitude toward variables is that a "*variable*" is a letter or symbol. Associated with each variable there is a window box. On the cover, the associated variable is engraved. Inside the box is a number. The variable on the cover is to be regarded as *a name for the number inside the box*. The number inside is the "*value*" of the variable. This value changes from time to time. "*Assignment*" is the process of giving a new value to a variable. In this process, we destroy the old value and put in the new one. When we wish to make a computation with the value of a variable, we "*read*" the value through the window. This reading does not alter the value of the variable. Once a variable is assigned a value, it will keep that value until some other value is specifically assigned to it.

WARNING TO THE TEACHER

Assignment is quite different from any concept the student will have met in mathematics. Because of its similarity to "equality" it is often mistaken as such. The use of the window boxes should avoid such a misconception on the part of the student.

Class Activity Exercise 3-4b, in the next section, will be of great value in firmly establishing the concepts of assignment and variable. *Be sure this activity takes place*.

Exercises 3-3a (Class Discussion)

1. (a)

	B	C
Starting values	9	4
Final values	4	4

$\boxed{B \leftarrow C}$

(b)

	B	C
Starting values	9	4
Final values	9	9

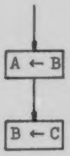
$\boxed{C \leftarrow B}$

(c) The final values of B and C are the same.

(d) Different. They assign different values.

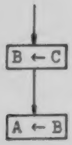
2. (a)

	A	B	C
Starting values	9	11	13
Values after first assignment	11	11	13
Final values	11	13	13



(b)

	A	B	C
Starting values	9	11	13
Values after first assignment	9	13	13
Final values	13	13	13



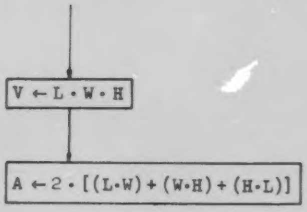
(c) In both cases the assignments are the same, that is, $\boxed{A \leftarrow B}$ and $\boxed{B \leftarrow C}$.

(d) The order in which the assignments are carried out has been interchanged.

(e) Yes. Compare the final values in the two tables.

Exercises 3-3b

	L	W	H	V	A
1.	7	3	2	42	82
2.	8	3.5	5	140	171
3.	11	9	7	693	478
4.	12	2.3	4.6	126.96	186.76


$V \leftarrow L \cdot W \cdot H$
$A \leftarrow 2 \cdot [(L \cdot W) + (W \cdot H) + (H \cdot L)]$

3-4. Input and Output

In this section we discuss two more basic kinds of procedural steps called input and output. "Input" and "assignment" statements are closely related in that both result in the assigning of values to the variables. Moreover, input is closely related to output in that one process can be thought of as the reverse of the other. Hence, all three concepts, input, output, and assignment, seem to be directly or indirectly related.

For clarification it should be noted that there are two kinds of input associated with the computing process--input of data, which is what we are discussing; and input of the program (i.e., the set of instructions or commands). Program input will not be discussed in this chapter.

The phrases "read a specific value into storage" and "write out a specific value from storage" are often used in speaking about input and output processes. Modern computers are equipped with a variety of input devices which can read data supplied through appropriate input media. Computers at banks, for example, have input devices which read account numbers printed on checks if these numbers are printed with a special magnetic ink. When typewriters are attached to computers, the data may be supplied simply by typing it. One of the most popular input devices is the card reader which reads punch cards, so you can see that it was natural to adopt as a flow chart convention the silhouette of a punch card to suggest the input process.

Exercises 3-4a (Class Discussion)

1. (a) Valid

(b) Invalid No computation should be indicated in an input box.

(c) Invalid Only variables should occur in an input box.

(d) Valid

(e) Valid

(f) Invalid No computation should be indicated in an input box.

(g) Invalid An assignment symbol should not occur in an input box.

2. (a) Valid

(b) Invalid No computation should be indicated in an output box.

(c) Invalid The variables should be separated by a comma in an output box.

(d) Invalid Only variables should occur in an output box.

(e) Invalid Neither computation nor assignment should occur in an output box.

(f) Invalid Only variables should occur in an output box.

(g) Valid

3. (a) The machine would accept the card, A, B, A , and read into storage, in order, the values assigned to the variables in the list. When the second value of A is read into storage the first value would be destroyed.

(b) B would have a value of 9.

A would have a value of 7.

4. (a) The computer would print out three values. The first and third values would be the same.

(b) The computer would print out 19, 11, 19 .

5. (a) Invalid — There is no assignment arrow and no arithmetic expression in the box.

   (b) Valid

   (c) Invalid — The arrow is pointing the wrong way. The arithmetic expression should be on the right-hand side of the arrow and the single variable on the left-hand side.

   (d) Valid — 2 is an arithmetic expression which can be assigned to the variable A.

   (e) Invalid — A single variable should be on the left-hand side of the arrow. "2" is not a variable.

   (f) Invalid — A single variable should be on the left-hand side of the arrow. "2" is not a variable.

   (g) Invalid — A single variable should be on the left-hand side of the arrow. "2" is not a variable.

   (h) Invalid — There is no assignment arrow given, and no single variable to assign the value of the arithmetic expression in the box.

   (i) Invalid — "B, C" is not an arithmetic expression.

   (j) Invalid — "B or C" is not an arithmetic expression.

   (k) Invalid — The equal sign, "=", should not occur in an assignment box.

   (ℓ) Valid

Exercises 3-4b (Class Activity)

The actual, physical simulation of the operation of a simple computer seems to be indispensable to the understanding of the ideas of assignment, variables, input, and output. You are strongly urged to introduce this model of a computer in your classroom. You may want to draw the window boxes on the board instead of using shoe boxes. The student then can show destruction of a value of a variable by erasing the value before he assigns it a new value. It might also be helpful to have the flow chart in Figure 8 on the board. The input and output values are listed below for your convenience.

Input		Output		
R	T	R	T	W
2.00	27	2.00	27	54.00
2.50	38	2.50	38	95.00
1.75	36	1.75	36	63.00
2.10	40	2.10	40	84.00
2.25	39	2.25	39	87.75
1.95	40	1.95	40	78.00

Exercises 3-4c

1.

(a)

R	T	W
2	27	54.00
2.15	39	83.85
1.87	41.75	78.0725
1.945	37.25	72.45125

(b)

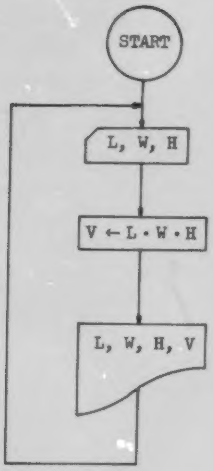
R	T	W
2	27	54.00
2.15	39	83.85
1.87	41.75	78.07
1.945	37.25	72.45

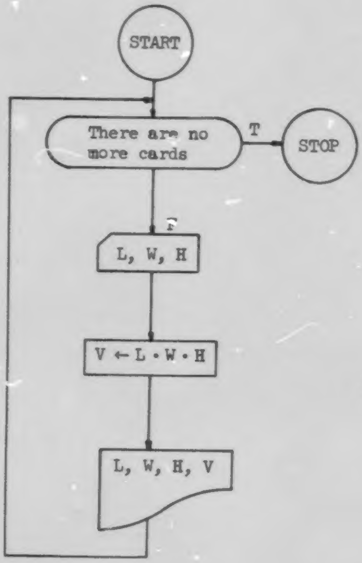
2.

	R	T	W
(a)	2	27	54
(b)	1.87	41.75	83.85
(c)	1.945	37.25	78.0725
(d)	2	27	No value known
(e)	No values known		

(**Note**: Discuss this problem carefully.)

3. This is the first flow chart that the students will construct for themselves involving a mathematical algorithm. Be prepared to accept some variation in their responses. The basic criteria for an acceptable flow chart is, "Does it work?" It might be informative to discuss several "student versions" in class. Here are two possible flow charts.





3-5. Using a Variable as a Counter

There are three basic ideas presented in this chapter.

First, we meet the situation where the new value of a variable is calculated in terms of the former value. If any of your students are still thinking of assignment in terms of "equality" here is an opportunity to clarify the meaning of such an assignment box. In the assignment $N \leftarrow N + 1$, it should be obvious that the value of N cannot be equal to one more than itself.

Second, the idea of the necessity of a variable having a starting value is brought out clearly in Class Discussion Exercises 3-5a. It is suggested that the teacher trace through the flow chart on the chalkboard or on the overhead projector as the students trace through it at their desks.

Finally, Figures 11a, 11b, and 12 are all examples of different flow charts of the same algorithm. Again, it can be pointed out to the students that the choice of which of the three flow charts is most desirable usually depends on which one is the simplest and most efficient, namely Figure 11.

Class Discussion Exercises 3-5b revises the flow chart of Figure 12 to incorporate an additional output. Since Exercise 3-5c, which follows, is dependent on the revised flow chart, the teacher should be certain that the students have developed it correctly, preferably in conjunction with the teacher's development.

Exercises 3-5a (Class Discussion)

1. The reason for this trace is that hopefully the students will arrive at the assignment $\boxed{N \leftarrow N + 1}$ and realize they cannot go any further, because N does not have a starting (initial) value.
2. Assign to N the value of N + 1.
3. No. The variable, N, has not been assigned an initial value.
4. An assignment box must be added to the flow chart that will assign an initial value to N.

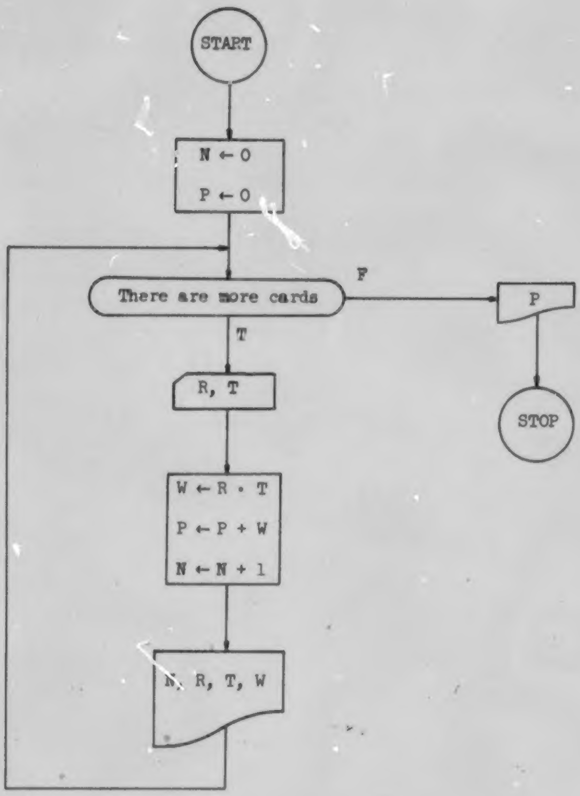
Exercises 3-5b (Class Discussion)

1. (a) $\boxed{P \leftarrow P + W}$

   (b) $\boxed{P \leftarrow 0}$

   (c) P will have the desired value when all the wages have been calculated and totaled.

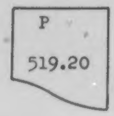
(d)


START
N ← 0
P ← 0
There are more cards
F
P
T
R, T
STOP
W ← R · T
P ← P + W
N ← N + 1
N, R, T, W

Exercises 3-5c

1.

	N	R	T	W
(a)	1	2.50	32	80.00
(b)	2	3.00	38	114.00
(c)	3	3.40	22	74.80
(d)	4	2.75	40	110.00
(e)	5	3.60	39	140.40



Note: In this problem it is desirable, but not necessary, that the students follow the above format.

3-6. Decision and Branching

The development of decision boxes that contain mathematical statements enables us to construct flow charts for algorithms of any degree of complexity. "Branching" gives us the ability to choose a new path depending on whether a certain condition is satisfied. A decision box always has two exits and it is in this way that it differs from all other boxes we have met in flow charting.

Exercises 3-6a (Class Discussion)

> **Note:** If your students are familiar with the distributive property, then you might want to discuss the simplification of the expression developed in this section. Don't overemphasize this simplification process as it will be treated at an appropriate time in a later chapter. The simplification steps are as follows:
>
> $$40 \cdot R + 2 \cdot R \cdot (T - 40)$$
>
> $$2 \cdot R \cdot (20 + (T - 40))$$
>
> $$2 \cdot R \cdot (T - 20)$$

1. $40 \cdot R$ or $40R$
2. $2 \cdot R$ or $2R$
3. $T - 40$
4. $2 \cdot R \cdot (T - 40)$ or $2R(T - 40)$
5. $40 \cdot R + 2 \cdot R \cdot (T - 40)$ or $40R + 2R(T - 40)$

Exercises 3-6b

1.

N	R	T	W
1	2.15	39	83.85
2	2.64	44	126.72
3	1.98	27	53.46
4	2.15	40	86.00
5	2.26	45	113.00

P

$\boxed{463.03}$

2. The order of two assignment statements may be interchanged if:

(a) the variables to the left of the arrows are different,

AND (b) the value of the variables on the right of the arrow in the second assignment is not changed by the first assignment statement.

(Note: Both conditions must be met.)

For example:

In the assignment box $\begin{array}{|l|}\hline A \leftarrow 2 \\ A \leftarrow 3 \\ \hline\end{array}$ the order of the assignments cannot be interchanged without altering the values assigned to A.

In the assignment box $\begin{array}{|l|}\hline L \leftarrow 2 \cdot A \\ A \leftarrow L \cdot W \\ \hline\end{array}$ the order of the assignments cannot be interchanged without altering the values assigned to the variables, L or A.

3. Output

N	T	S
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
7	13	49
8	15	64
9	17	81
10	19	100

4. (a) N has an initial value of 1. N increases by one each time through the output box until a value of 10 is reached.

(b) T has an initial value of 1. T is increased by 2 for each output until a final value of 19 is reached.

(c) After the first value of S, each value of S can be found by adding the current value of T to the previous value of S.

(d) The assignment command $S \leftarrow S + T$.

(e) Yes. Add the current value of T to all of the previous values of T to get the current value of S.

(f) Sum, T.

5.

Output	N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	S	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

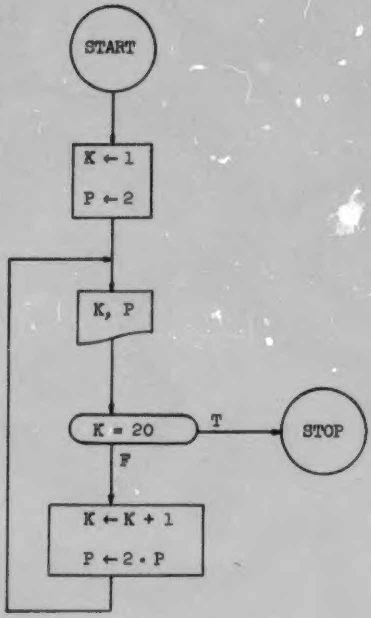
6. (a) Each value of K is obtained by increasing the previous value of K by 1.

(b) Each value of P is obtained by multiplying the preceding value of P by 2.

(c)

K	7	8	9	10	11	
P	128	256	512	1024	2048	

(d)


START
K ← 1
P ← 2
K, P
K = 20
T
STOP
F
K ← K + 1
P ← 2 · P

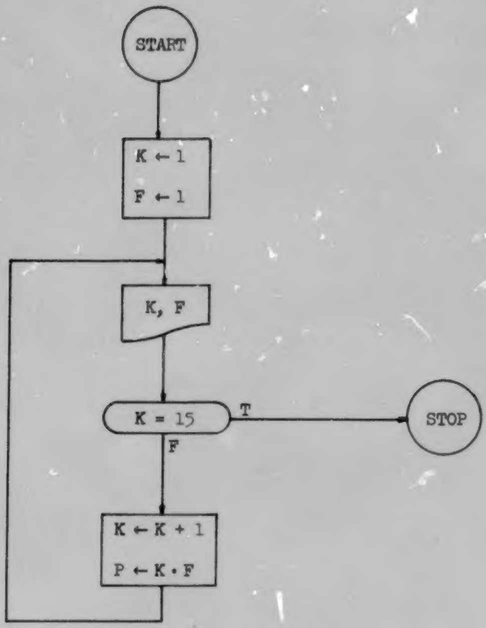
7. (a) Each value of K is obtained by increasing the preceding value of K by 1.

(b) Each value of F is obtained by multiplying the preceding value of F by the current value of K.

(c)

K	6	7	8
F	720	5040	40,320

(d)



Exercises 3-6c (Class Discussion)

1. John, George, Tom, and Gordy had correct solutions.

2. Paul's solution is incorrect because his outputs would be numbered starting with 0 instead of 1.

   Pete's solution is incorrect because his outputs would not be numbered consecutively.

   Lars' solution is incorrect because he would be able to input only one card.

   Bob's solution is incorrect because he could only get one output.

3. Tom's is the most efficient flow chart.

Gordy and John are next since they will do fewer computations than George because an input will be rejected if the perimeter is less than or equal to 12, and the area computation will not be done.

George's is the most inefficient because he will compute the area for all input values of L and W.

3-7. Flow Charting the Division Algorithm

This section uses flow charts to analyze the process of division. In this particular case, division is presented as a form of "repeated subtraction."

The flow chart for division by 7 is developed primarily to enable the student to easily make the transition to a flow chart using a general divisor. In making this transition, the student needs only to replace the 7 with the variable D. The development and use of the flow chart of the integer division algorithm should enable the student to improve his understanding of the division process. In addition, the development provides one model for transferring from the particular to the general.

Exercises 3-7a (Class Discussion)

1.

	Q	R
(a)	0	17
(b)	1	10
(c)	2	3
(d)	2	3
(e)	2	3

Exercises 3-7b

1. Follow the flow chart of Figure 15. Use any input for M and let D = 0. In the assignment box $\boxed{\begin{array}{l} R \leftarrow R - D \\ Q \leftarrow Q + 1 \end{array}}$ R is assigned the value R - 0, and Q is increased by one. $R > D$ for every step. The loop will continue indefinitely, R will remain the same, and Q will increase indefinitely. You therefore find yourself caught in an endless loop from which there is no way out. **Note**: Here is a visual representation of why division by zero is meaningless and this fact should be emphasized.

2. (a) Done in the text.

(b)

	M	D	Q	R
1st	24	5	0	24
2nd	24	5	1	19
3rd	24	5	2	14
4th	24	5	3	9
5th	24	5	4	4

(c)

	M	D	Q	R
1st	5	24	0	5

(d)

	M	D	Q	R
1st	0	23	0	0

(e)

	M	D	Q	R
1st	24	8	0	24
2nd	24	8	1	16
3rd	24	8	2	8
4th	24	8	3	0

(f)

	M	D	Q	R
1st	73	6	0	73
2nd	73	6	1	67
3rd	73	6	2	61
4th	73	6	3	55
5th	73	6	4	49
6th	73	6	5	43
7th	73	6	6	37
8th	73	6	7	31
9th	73	6	8	25
10th	73	6	9	19
11th	73	6	10	13
12th	73	6	11	7
13th	73	6	12	1

3. At no place in the flow chart are there assignments to either M or D that would change their input values.

4.

	Q	R
(a)	2	3
(b)	3	0
(c)	24	1
(d)	6	0
(e)	1	0
(f)	0	17
(g)	3	2
(h)	1	0
(i)	3	2
(j)	7	0
(k)	14	0
(ℓ)	62	0
(m)	17	0
(n)	452	4
(o)	4	14

5. (a) 1 and 0.

   (b) M and 0.

   (c) No. Since $M = D \cdot Q + R$, if only Q and R are given there is not enough information to determine M and D.

The following is a more practical flow chart development for the division algorithm incorporating the ideas of our place value notation. Because development of the abbreviated division algorithm is so time consuming, we have omitted it from the student text. We have included it in the teacher's commentary in anticipation that students may wish to investigate the division process more thoroughly.

If time permits, it may be of value to reproduce both the flow chart and the trace and go over the process in class. It can then be pointed out how closely the flow chart describes the "escalator" or "ladder" method of division learned in elementary school.

Making the Division Algorithm Practical

In the preceding section we constructed a flow chart for an integer division algorithm. This algorithm is perfectly correct as it always produces the right answer, but it is rather inefficient. For example, if we use this algorithm to divide 3168 by 7, our input values of $M = 3168$ and $D = 7$ yield output values of $Q = 452$ and $R = 4$. The trouble is that to obtain these outputs by using our algorithm requires 7 to be subtracted 452 times!

$$\begin{array}{r} 3168 \\ \underline{\quad 7} \\ 3161 \\ \underline{\quad 7} \\ 3154 \\ \underline{\quad 7} \\ 3147 \\ \text{etc.} \end{array}$$

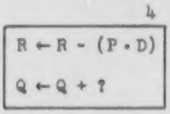
This is extremely time consuming and you know we would not do it that way. The thing that is poor about our algorithm is that it does not take advantage of the short cuts offered by our place value notation. In our base 10 (or decimal) notation, multiplying by 10 is especially easy. Each time we need to multiply a whole number by 10, we merely tack on a zero at the end of the numeral.

We use this idea in division in the following way. First we multiply our divisor by a power of 10, namely the largest possible power of 10 that gives a product no larger than our dividend:

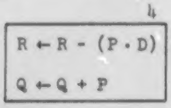
$70 < 3168$? Yes.
$700 < 3168$? Yes.
$7000 < 3168$? No.

Thus $700 = 7 \cdot 100$ is the largest such product. Subtracting this number from our dividend just once is equivalent to subtracting the divisor 7 a hundred times. An obvious short cut!

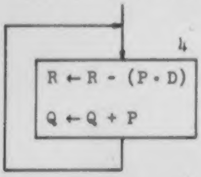
In flow charting our improved algorithm we must remember that each time we subtract $7 \cdot 100$ from our remainder we must add 100 to our quotient. Let us try to write a flow chart that includes our idea for a short cut. Beside all the same variables as before, M, D, Q, R, we will have a new one, P, that is the appropriate power of 10. That is to say, P takes on values such as 1, 10, 100, 1000, etc. And instead of decreasing the value of R by D we will in our short cut decrease the value of R by P • D.



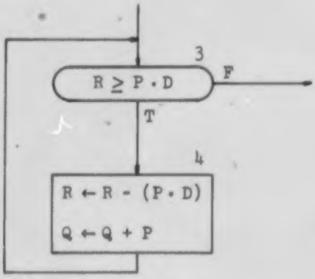
What should we do to the value of Q? Well, you must remember that Q is just keeping count of the number of times D is subtracted. Since D has been subtracted P times, the value of Q must therefore be increased by P to keep the count correct.



We repeat this process as many times as we can.



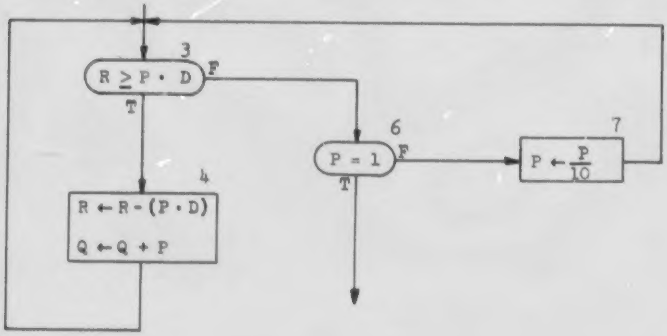
By "as many times as we can" we mean "as long as R ≥ P • D". A decision box is necessary to tell us when to leave the loop.



This is the basic mechanism in our flow chart.

The next step is to reduce the value of P and go back into the loop of boxes 3 and 4. How does P get its new value? The answer is: by dividing the old value by 10. (Here we must assume that the computer knows how to perform such elementary divisions as dividing 10 or 100 or 1000 by 10, which merely consists of lopping off a zero.) Of course, this cannot be done once P has already obtained its lowest permissible value of 1.

With these features our flow chart looks like this:

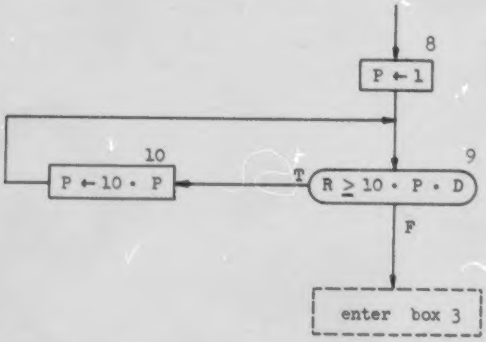


Finally we must find how P gets the value with which it first enters box 3. P must be the largest power of 10 (1 or 10 or 100 or 1000, etc.) such that $R \geq P \cdot D$. We get this value by starting P with the value 1 and stepping it up through the values 10, 100, 1000, etc., until the next larger power of 10 (that is, $10 \cdot P$) would not work, that is, until the statement

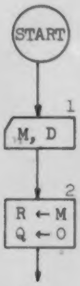
$$R \geq 10 \cdot P \cdot D$$

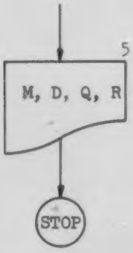
is false.

A flow chart fragment for accomplishing this is:

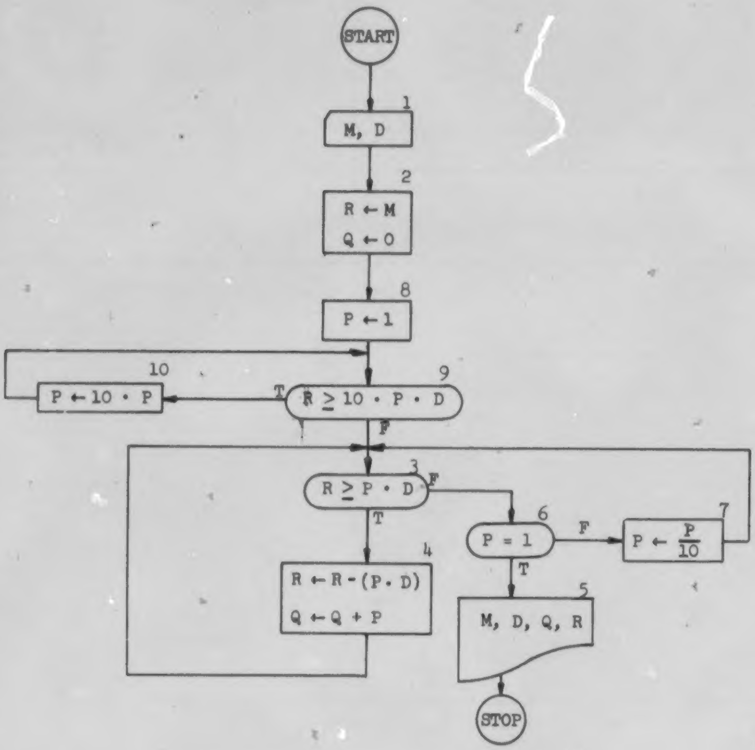

8
P ← 1
10
P ← 10 · P
T
9
R ≥ 10 · P · D
F
enter box 3

All that remains is to put on the beginning and end of the flow chart.


START
1
M, D
2
R ← M
Q ← 0



Assembling the pieces we at last have completed our flow chart for efficient integer division.


START
1
M, D
2
R ← M
Q ← 0
8
P ← 1
10
P ← 10 · P
9
T
R ≥ 10 · P · D
F
3
R ≥ P · D
F
T
4
R ← R - (P · D)
Q ← Q + P
6
P = 1
F
7
P ← P/10
T
5
M, D, Q, R
STOP

Efficient Integer Division

Trace of Flow Chart for M = 3168, D = 7

Step	Flow Chart Box	Values of Variables				Decision
			Q	R	P	
1	1	M = 3168, D = 7				
2	2		0	3168		
3	8				1	
4	9					3168 ≥ 70 T
5	10				10	
6	9					3168 ≥ 700 T
7	10				100	
8	9					3168 > 7000 F
9	3					3168 ≥ 700 T
10	4		100	2468		
11	3					2468 ≥ 700 T
12	4		200	1768		
13	3					1768 ≥ 700 T
14	4		300	1068		
15	3					1068 ≥ 700 T
16	4		400	368		
17	3					368 ≥ 700 F
18	6					100 = 1 F
19	7				10	
20	3					368 ≥ 70 T
21	4		410	298		
22	3					298 ≥ 70 T
23	4		420	228		
24	3					228 ≥ 70 T
25	4		430	158		
26	3					158 ≥ 70 T
27	4		440	88		
28	3					88 ≥ 70 T
29	4		450	18		
30	3					18 ≥ 70 F
31	6					10 = 1 F
32	7				1	
33	3					18 ≥ 7 T
34	4		451	11		
35	3					11 ≥ 7 T
36	4		452	4		
37	3					4 ≥ 7 F
38	6					1 = 1 T
39	5	M = 3168, D = 7	452	4		

Suggested Test Items

Multiple Choice

1. Which of the following is not a valid assignment box?

(A) $\boxed{T \leftarrow P}$ (B) $\boxed{3 \leftarrow R}$ (C) $\boxed{S \leftarrow S}$

(D) $\boxed{T \leftarrow R \cdot M}$ (E) $\boxed{W \leftarrow 2}$

2. In the assignment box $\boxed{B \leftarrow C}$ the symbol $B \leftarrow C$ means,

(A) assign to B the value of C.

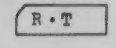
(B) assign to C the value of B.

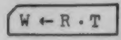
(C) B is equal to C.

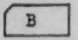
(D) B and C have the same value.

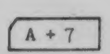
(E) none of these.

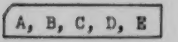
3. Which of the following are valid input boxes?

(A)


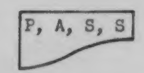
(B)


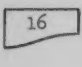
(C)


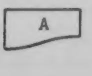
(D)


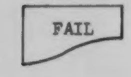
(E)


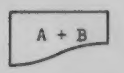
4. Which of the following are valid output boxes?

(A)


(B)


(C)


(D)


(E)


5. An algorithm is

(A) one of the symbols used in flow chart diagrams.

(B) a flow chart diagram.

(C) another name for a decision box.

(D) a list of instructions for carrying out a process.

(E) a division problem.

6. In computer work a variable is

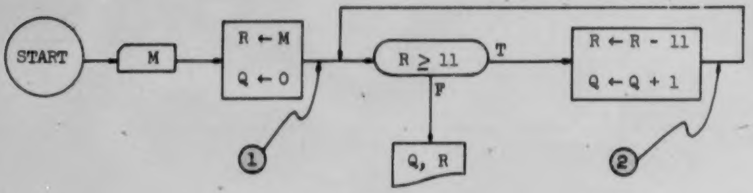
(A) an operation used in an assignment box.

(B) the memory device of the computer.

(C) a set of instructions for drawing a flow chart.

(D) a specific number that never changes its value.

(E) a letter that represents a number.



The figure above is a flow chart of the Integer Division Algorithm. The divisor in this problem is 11. Answer problems 7-10 using this flow chart.

7. If the dividend is 56, then what are the values of R, M, and Q the first time you pass ① ?

(A) 56, 56, 56 (B) 0, 0, 0 (C) 56, 0, 0

(D) 56, 56, 0 (E) 0, 56, 56

8. If the dividend is 56, then what are the values of R and Q the first time you pass ② ?

(A) 11, 1 (B) 45, 1 (C) 45, 0

(D) 56, 0 (E) None of these

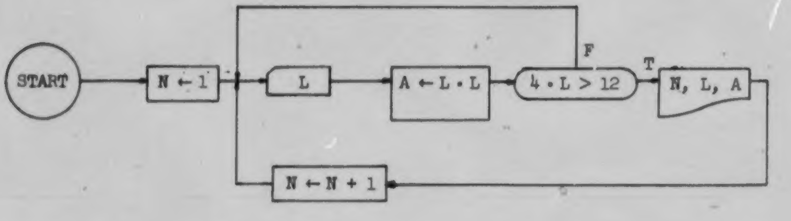
9. If the dividend is 67, then what are the values of R and Q the third time you pass ② ?

(A) 34, 3 (B) 23, 3 (C) 45, 3

(D) 56, 3 (E) None of these

10. Find the output Q, R when M is 81.

(A) 8, 0 (B) 7, 4 (C) 6, 5 (D) 5, 3 (E) 4, 4



The above diagram is a flow chart for finding the area of square regions given the lengths of the side L. Problems 11-15 refer to the above flow chart.

11. What is the output if the input L is 3?

(A) 1, 3, 9 (B) 1, 3, 12 (C) 2, 3, 9

(D) 2, 3, 12 (E) None of these

12. What is the output if the input L is 5?

(A) 1, 5, 25 (B) 1, 5, 20 (C) 2, 5, 25

(D) 2, 5, 20 (E) None of these

13. What was the input if the output for A was 81?

(A) $40\frac{1}{2}$ (B) 9 (C) 36

(D) 81 (E) None of these

14. Input the following three values for L: 4, 8, and 16, and calculate the corresponding outputs. How are the outputs for A related?

(A) No relationship.

(B) The value of A is doubled when L is doubled.

(C) The value of A is tripled when L is doubled.

(D) The value of A is multiplied by 4 when A is doubled.

(E) None of these.

15. If the input for L was: 1, 2, 3, 4, 5, then the last output would be

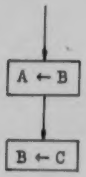
(A) 5, 5, 25 (B) 3, 5, 25 (C) 2, 5, 25

(D) 1, 5, 25 (E) None of these

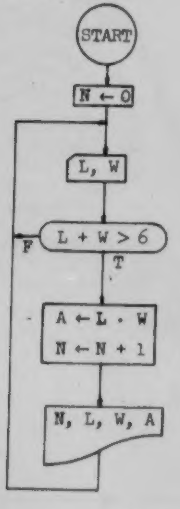
Problems

The initial values of A, B, and C are given in the table below. Follow the instructions indicated by the flow chart fragment on the right. Fill in the values in the table.

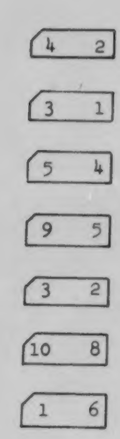
		A	B	C
	initial values	10	15	20
1.	values after first assignment			
2.	final values			



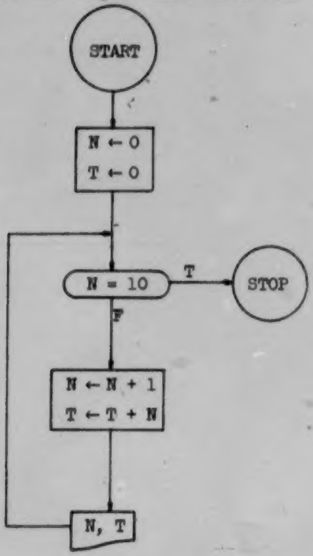
USE THE FLOW CHART BELOW TO PROCESS THE DATA INDICATED ON THE PUNCH CARDS DISPLAYED AT THE RIGHT. COMPLETE THE TABLE SO THAT EACH ROW REPRESENTS THE PRINT OUT FOR THE RELATED DATA. IF THERE IS NO PRINT OUT FOR ANY INPUT, STATE THIS FACT.



3.				
4.				
5.				
6.				
7.				
8.				
9.				



Trace through the flow chart below using the table to the right to list the output values each time through the output box.



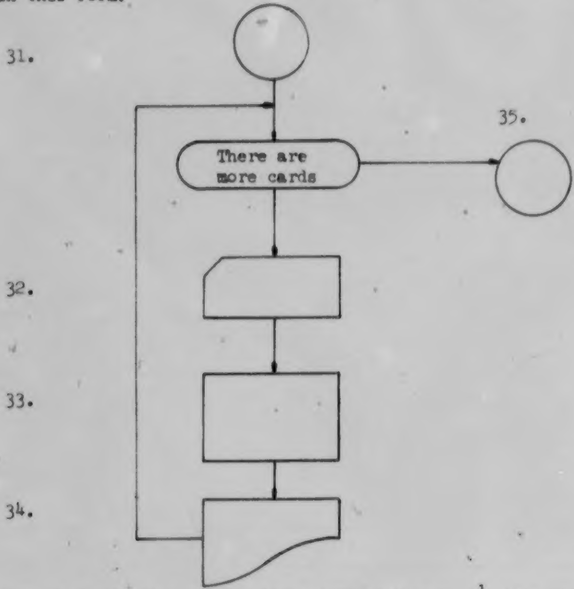
	N	T
10.		
11.		
12.		
13.		
14.		
15.		
16.		
17.		
18.		
19.		
20.		

In the space below, copy the above flow chart but change the order of the assignments. Trace through this new flow chart using the table to the right to list the output values each time through the output box.

	N	T
21.		
22.		
23.		
24.		
25.		
26.		
27.		
28.		
29.		
30.		

In a previous chapter, you learned that in the "falling body" function an input of t (seconds) outputs $16t^2$ (feet), that is, $f : t \to 16t^2$. The formula, which serves to specify this function, is expressed as $D = 16t^2$, where D is the distance traveled and t is the time spent in falling.

Below is a flow chart of this formula with the necessary information missing from each box. You are to supply the variables, commands, expressions, or decisions that belong in each box in order to flow chart this formula. Be sure you output all the variables, and remember, the computer cannot compute t^2 in this form.



36. The area of a triangle can be found by multiplying $\frac{1}{2}$ the number of units in the base by the number of units in the altitude. Thus, a formula for finding the area of a triangle can be specified as:

$$A = \frac{1}{2} b \cdot a \qquad \text{or} \qquad A = \frac{b \cdot a}{2} .$$

Construct a flow chart that inputs the number of units in the base (b), the number of units in the altitude (a), and outputs the area (A). The purpose is to produce an output with consecutively numbered lines giving the base, altitude, and area of only those triangles having an area greater than 7 square units. (Hint: It will help if you construct a table with columns for N, a, b, and A.)

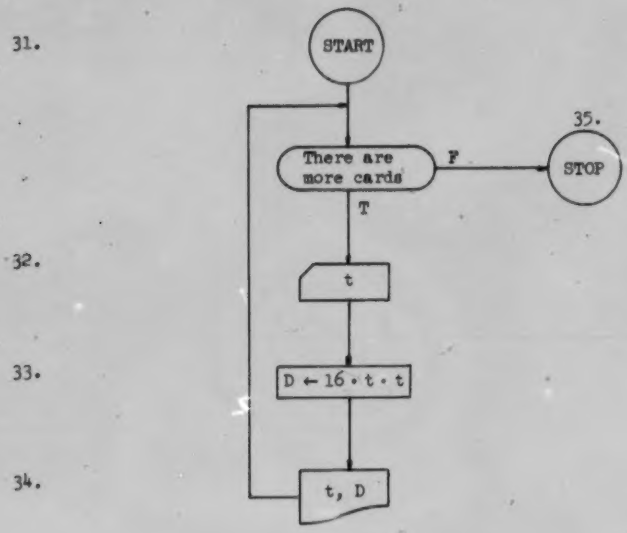
Answers to Suggested Test Items

Multiple Choice

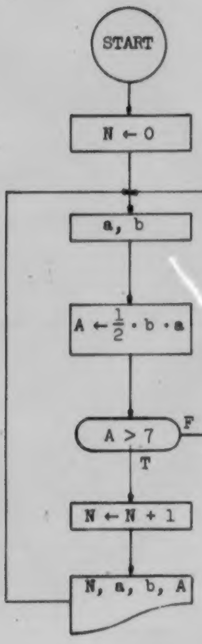
1. B
2. A
3. C, E
4. A, C
5. D
6. E
7. D
8. B
9. A
10. B
11. E
12. A
13. B
14. D
15. C

Problems

1. 15 15 20
2. 15 20 20
3. No print out
4. No print out
5. 1, 5, 4, 20
6. 2, 9, 5, 45
7. No print out
8. 3, 10, 8, 80
9. 4, 1, 6, 6
10. 1, 1
11. 2, 3
12. 3, 6
13. 4, 10
14. 5, 15
15. 6, 21
16. 7, 28
17. 8, 36
18. 9, 45
19. No print out
20. No print out
21. 1, 0
22. 2, 1
23. 3, 3
24. 4, 6
25. 5, 10
26. 6, 15
27. 7, 21
28. 8, 28
29. 9, 36
30. No print out


31.
START
32.
33.
34.
35.
There are more cards
F
STOP
T
t
D ← 16 • t • t
t, D

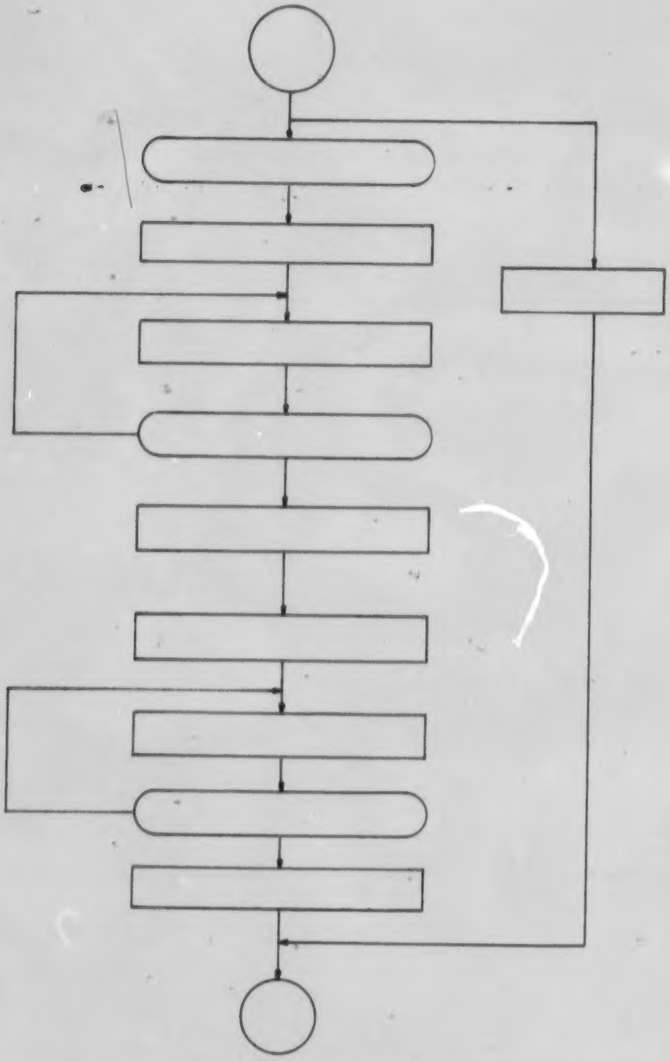
36.


START
N ← 0
a, b
A ← 1/2 • b • a
A > 7
F
T
N ← N + 1
N, a, b, A

To the Teacher

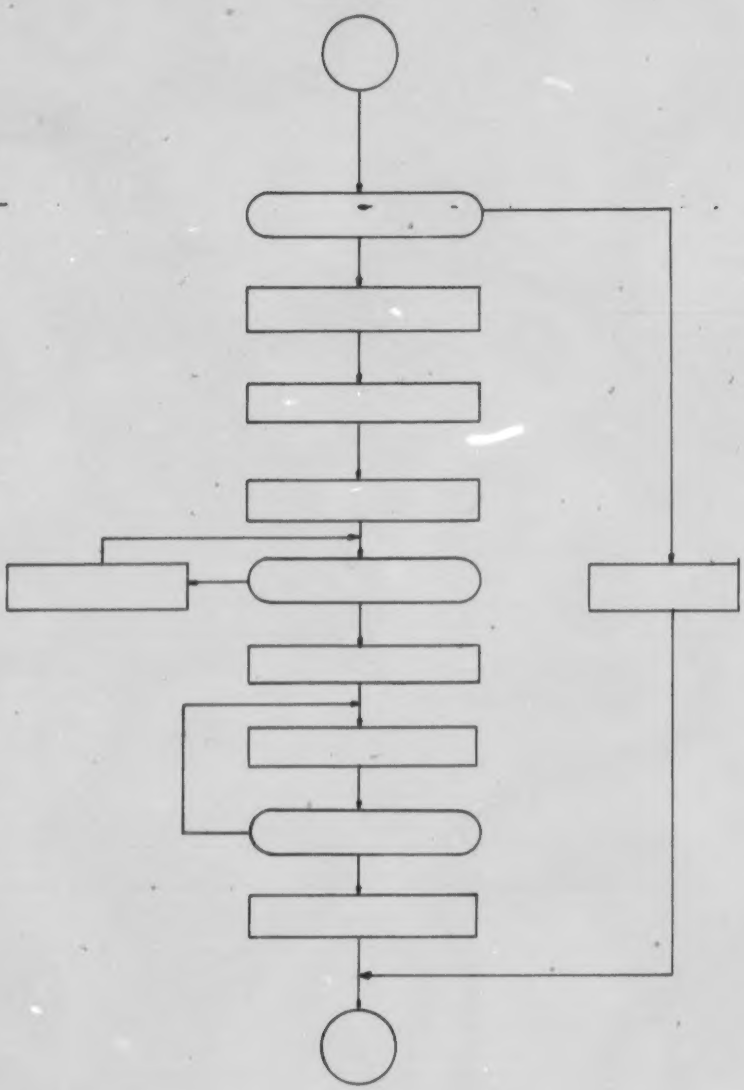
The following 8 pages contain the flow charts or the flow chart silhouettes of the more important algorithms covered in this chapter. These pages can be removed and either dittoes or overhead transparencies made from them. This should provide the means for the student to easily develop the flow chart at his desk as the teacher develops it on the chalkboard or overhead projector.

TEAR SHEET



Final flow chart for changing a flat tire

TEAR SHEET



Final flow chart for frying an egg

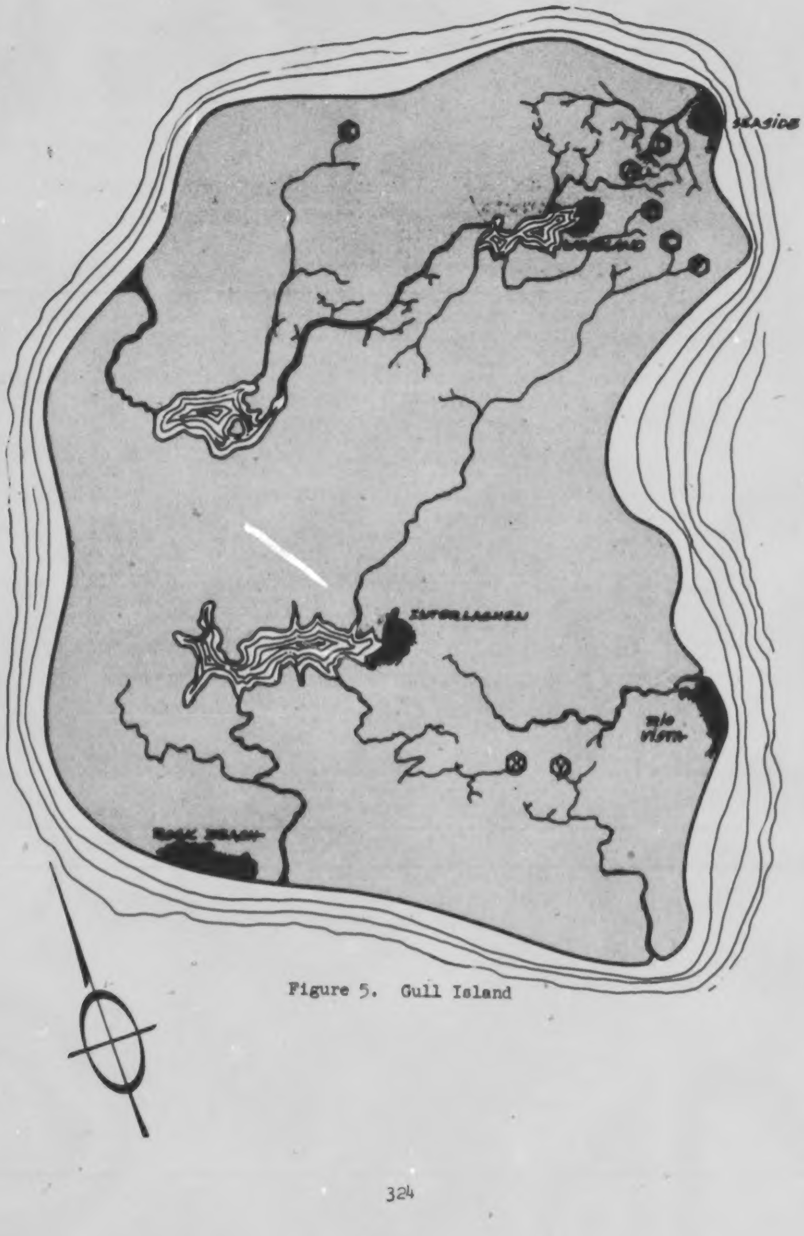
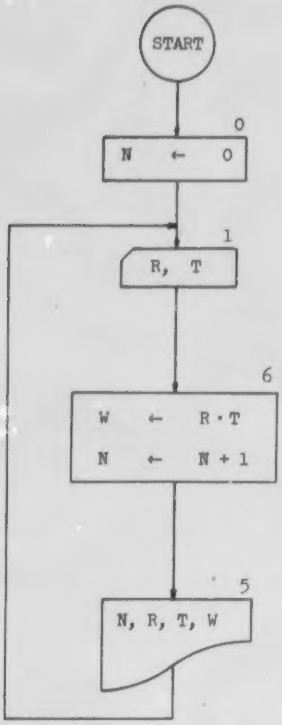

SEASIDE
INTERLAKEN
RIO VISTA
ROCK BEACH

Figure 5. Gull Island

TEAR SHEET


START
0
N ← 0
1
R, T
6
W ← R·T
N ← N + 1
5
N, R, T, W

Using a variable as a counter

TEAR SHEET

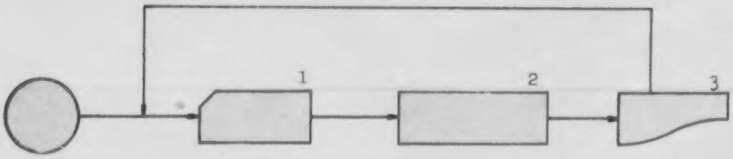
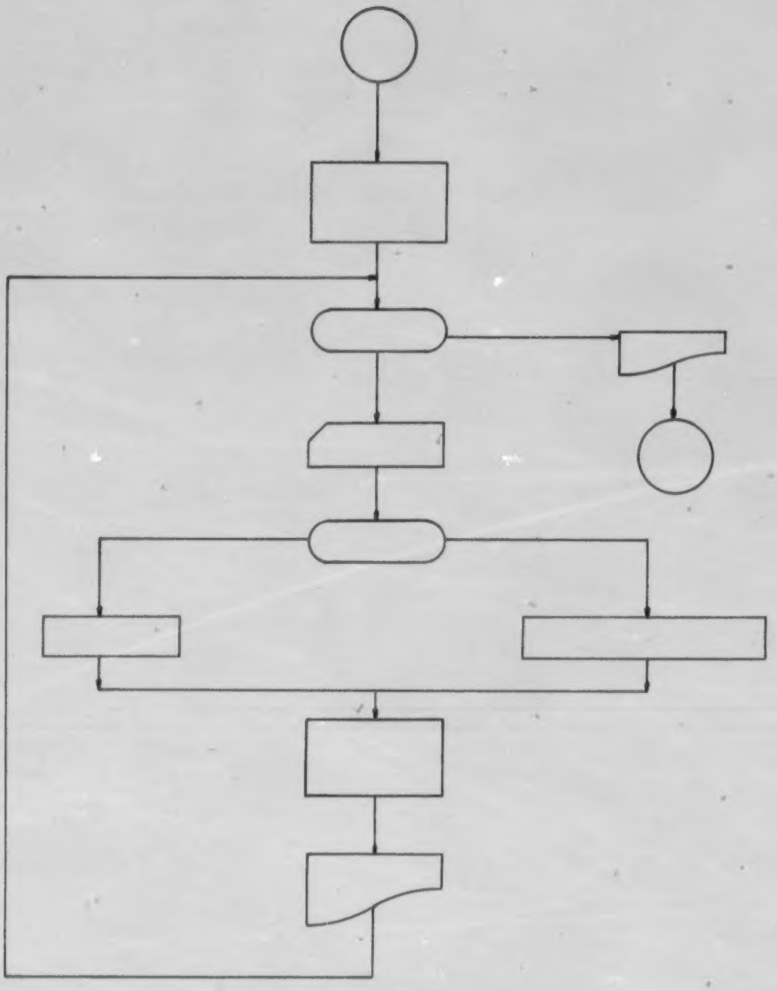
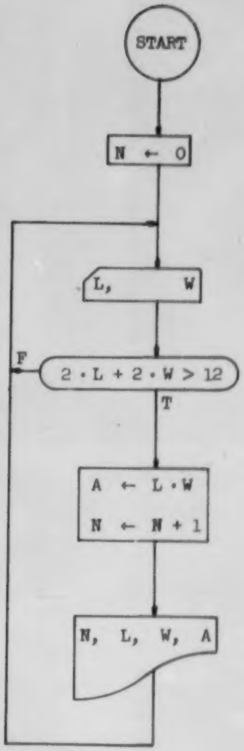


Figure 9

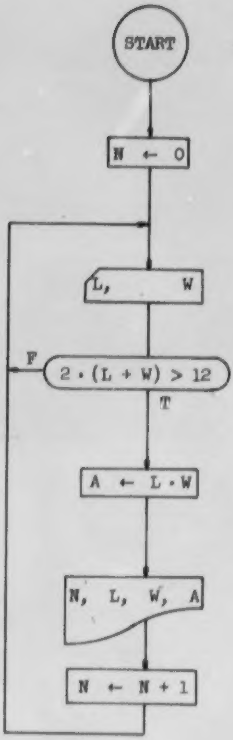
TEAR SHEET



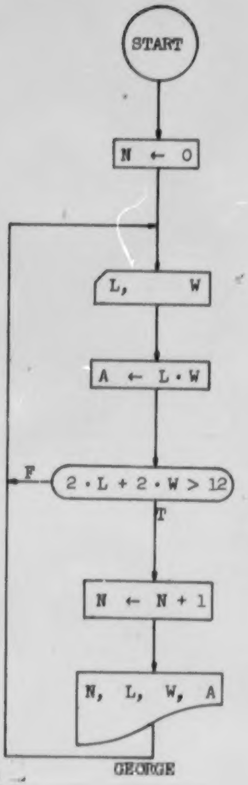
Flow chart for payroll, including double rate for overtime


START
N ← 0
L, W
F
2 • L + 2 • W > 12
T
A ← L • W
N ← N + 1
N, L, W, A

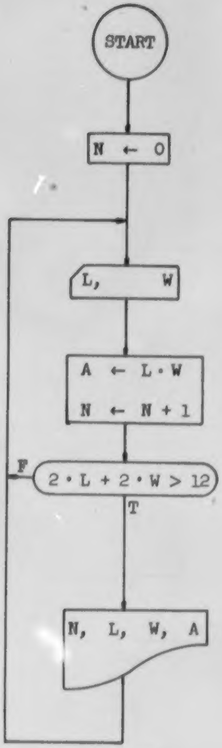
JOHN


START
N ← 0
L, W
F
2 • (L + W) > 12
T
A ← L • W
N, L, W, A
N ← N + 1

PAUL

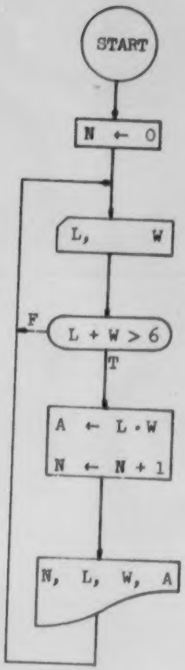

START
N ← 0
L, W
A ← L • W
F
2 • L + 2 • W > 12
T
N ← N + 1
N, L, W, A

GEORGE

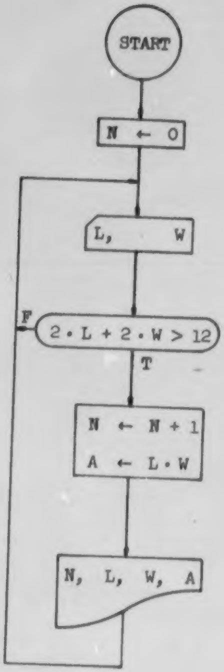

START
N ← 0
L, W
A ← L • W
N ← N + 1
F
2 • L + 2 • W > 12
T
N, L, W, A

PETE

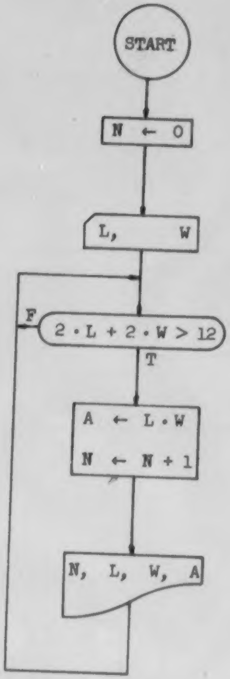
TEAR SHEET


START
N ← 0
L, W
F
L + W > 6
T
A ← L • W
N ← N + 1
N, L, W, A

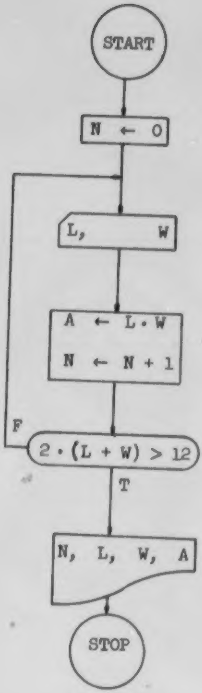
TOM


START
N ← 0
L, W
F
2 • L + 2 • W > 12
T
N ← N + 1
A ← L • W
N, L, W, A

GORDY


START
N ← 0
L, W
F
2 • L + 2 • W > 12
T
A ← L • W
N ← N + 1
N, L, W, A

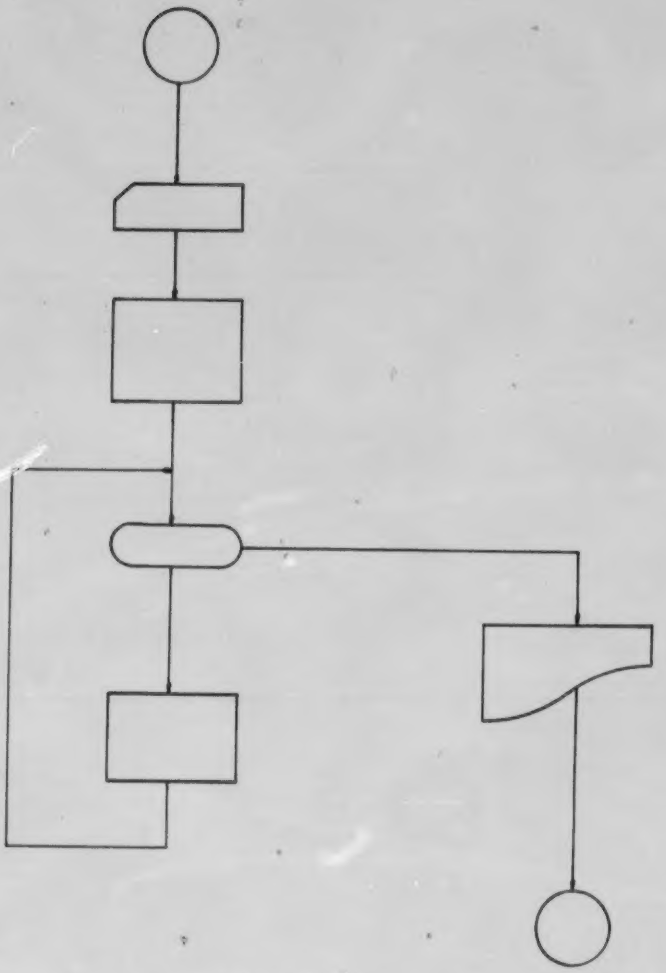
LARS


START
N ← 0
L, W
A ← L • W
N ← N + 1
F
2 • (L + W) > 12
T
N, L, W, A
STOP

BOB

TEAR SHEET

TEAR SHEET



Integer division

Chapter 4

APPLICATIONS AND MATHEMATICAL MODELS

4-1. Introduction

Mathematics does not deal directly with physical objects. Mathematics does talk about ideas such as points, lines, numbers, and functions. If these two statements are both true, then how can you use mathematics to solve problems dealing with real life objects or situations? The following story, told by a mathematician, may help you see the answer to this question.

> "On a recent trip to New York, I got off at Pennsylvania Station. I walked to the taxi platform, but the train had been crowded and soon dozens of people poured out of the station for cabs. Some of them waved and yelled; some stepped in front of the cabs; some, who had employed porters, seemed to be getting special treatment. I waited on the curb, convinced that if I acted in a civilized, proper manner I would attract a cab driver. However, when a driver stopped, he was snatched out from under me. I finally gave up, took the subway, denouncing the railroad, the cabbies, and people in general, and hoping that they would all get stuck for hours in the crosstown traffic.
>
> "Several weeks later, on a trip to Philadelphia, I got off at the 30th St. Station. I walked to the taxi platform. A sign told me to take a number. A man in charge loaded the cabs in numerical order, and I was soon on my way to the hotel. It was quick, it was pleasant, it was civilized. This was an example of a fine, though very simple way in which mathematics can affect social affairs. The rule for loading was the order of arrival on the platform. The numbers have an order, and were used as a model of people standing in line without the inconvenience and indignity of standing in a line. The numbers were used to help turn raving madmen into polite human beings."[1]

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[1] From an address delivered by Philip J. Davis at the Annual Dinner of the Society of the Sigma Xi, Auburn University, Auburn, Alabama, May 23, 1962.

One way in which mathematics is used in the solution of real world problems is indicated in the above story. In order to solve problems about real life objects we usually create a "Mathematical Model" in which the real life objects are represented as mathematical objects. It is the purpose of this chapter to review and present some ways in which mathematical models play a part in applying mathematical ideas to the real world.

Let us begin with the uses made of the so-called *natural numbers*: 1, 2, 3, ... These numbers are also called *counting numbers*. You can easily imagine situations in which even quite primitive peoples would have a need for something like counting. It is also easy to imagine that once this dive into mathematics had been taken such things as "addition" of counting numbers would be invented to describe something about what happens when two sets of things are combined and that some early mathematical genius might discover, for example, the commutative property of addition of counting numbers. Our main point here is that what you have done for some time in applying numbers and geometry is not very different in basic spirit and method from what is done by men who write about how to use mathematical models to solve problems in business or an Einstein who seeks mathematical models for the functioning of things in the universe. (Working out mathematical models for such problems usually requires more mathematical knowledge than you have now, of course.)

With these things in mind, please consider the following exercises.

Exercises 4-1a

(Class Discussion)

In each of the following exercises you are given a mathematical model. Try to imagine and describe a real situation that can be represented by the given mathematical model.

Example: Given the set of natural numbers and the operation of addition. Describe a situation that is represented by the above model.

Answer: The total number of students in a mathematics classroom where there are 16 boys and 15 girls.

(The specific mathematical model of this situation would be $(16 + 15)$.)

1. Give examples of situations in the world for which the mathematical model produced in the world of mathematics involves addition, or multiplication, the set of natural numbers, and one of the relations =, <, or >.

2. Give a situation using addition where the numbers that represent things in the real world are natural numbers but where actual counting would probably not be possible.

3. Give an example of a situation in the world for which the mathematical model would involve fairly small natural numbers and subtraction. Now think of a different kind of situation for which the mathematical model would be exactly the same.

4. Give an example of a situation for which the mathematical model would involve natural numbers and division. Now try to think of a quite different situation which would have the same mathematical model.

5. Give a situation which would be described by natural numbers in the real world but where counting of individual objects probably did not lead to the numbers.

6. Give a situation where the initial description is in terms of natural numbers but where manipulation in the world of mathematics forces one to consider fractions.

7. Give a situation for which the appropriate mathematical description involves negative integers.

8. Give a situation from the real world for which the mathematical model would involve both geometry and arithmetic.

9. Give a situation where the appropriate mathematical model involves an equation, or a function.

One advantage of describing real situations by numbers is that numbers are "portable"--that is "2" is easier to move around from one place to another than *two cows*. Numerical information can be sent over telephone wires, used in computations done on paper, or easily "fed into" a computer. Another advantage is that the *same* numerical expression might serve as a mathematical model for many *different* kinds of situations in the real world.

In the first part of this section you found a situation or problem which could be represented by a given mathematical model. In the following exercises, you will be given a problem situation and asked to suggest a mathematical model which could represent the situation.

Exercises 4-1b

(Class Discussion)

1. Suppose that you're standing on a spot in a room and could shoot anybody in the room with your water pistol. Are there any room shapes which would prevent you from doing this?
2. Suppose that you are the manager of a baseball team. You need a new shortstop. You can trade for Willie Much or Mickey Little, both of whom appear to be equally good glove men. In previous play Willie Much has come to bat 225 times and has 53 hits. Mickey Little has come to bat 183 times and has 43 hits. On the basis of this information which would you choose?
3. A T.V. antenna wire enters a room at one of the corners formed by two walls and the ceiling. The owner of the house wanted the wire to run down through a wall and under the floor to the opposite corner formed by two walls and the floor. What is the shortest length of wire he can use?
4. Suppose that you go to a picnic and are invited to join either of two tables. At table A there are now sitting 7 people with 5 quarts of ice cream. At table B there are sitting 10 people with 7 quarts of ice cream. At which table will your share of ice cream be greater?

The previous exercises may have given you some understanding of the reasons why people are encouraged to learn a good deal of mathematics. It turns out that mathematics is not only convenient, but very useful in dealing with real world events and problems. This is true even though mathematics itself is made up of things that are not "real" at all, at least not in the sense that molecules, cows, bacertia, rockets, bridges, etc., are "real". The discovery of a good mathematical model for a given situation is an interesting but challenging process.

4-2. Situations Leading to Geometric Models

In Chapter 1 you represented mathematical ideas, such as lines, rays, planes, and angles, with drawings that were referred to usually as geometric figures. Such figures and their characteristics serve as a rich source of mathematical models of real life situations.

In Chapter 2, you considered the problem of falling objects and Galileo's experiment. Now we want to take a look at the modeling process involved in these chapters.

We made the following assumptions when talking about Galileo's experiment:

(1) We thought of the falling objects as points. (More accurately we thought of the locations of the objects as being points.)

(2) We thought of the surface of the earth as a plane, and we thought of the paths of the objects as being parallel line segments, both perpendicular to the earth's surface.

(3) We neglected air resistance, that is, we said the objects were falling in a vacuum, (no air).

(4) We assumed that the height that an object is dropped from has no effect on how far it travels in one second.

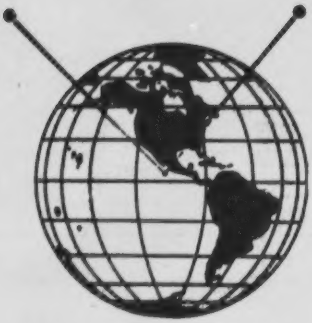
(5) Finally we assumed that the distance traveled by a falling object is given by the formula:

$$d = 16t^2$$

where t is the time in seconds and d is the distance in feet.

This is a rather strange picture of the world! The earth is a plane with no air above it and a falling object is squeezed down into a single point.

In fact, every one of our assumptions is wrong. We know that the earth is roughly spherical in shape and that falling objects will fall toward the center of the earth and their paths will not be parallel.



Furthermore, the distance traveled by a falling body in one second is not independent of the height of the starting point. Even if we neglect the effect of air resistance, an object falling from a mile high will fall less far in a second than an object dropped near the earth's surface. The amount less would be about one part in 2000. (The change in the pull of gravity due to the distance above the earth is responsible for the difference mentioned here.)

Air resistance is certainly not always negligible. It is because of air resistance that a piece of paper falls more slowly than a penny.

All of these remarks must have weakened your confidence in our model. That was what they were suppose to do. Now we are going to rebuild your confidence.

Although the earth is a sphere, it is such a big sphere that a small portion of its surface is very nearly a plane. If two objects are dropped so that they land no more than 100 feet apart, then their paths miss being parallel by about $\frac{1}{3500}$ of one degree, which is practically negligible.

The effect of the height of the starting point only produces a difference of one part in 2000 for objects dropped from a mile high. The effect will be even more negligible if we consider only objects dropped from within a few hundred feet of the earth's surface.

The effect of air resistance is very complicated. Its effect depends on the weight, shape, and speed of the falling body. For objects that are nearly round, fairly heavy, and falling for no more than two or three seconds (so as not to build up too much speed), we can comfortably neglect air resistance.

So our model for the motion of falling bodies is not so bad after all. In fact this model is used for very accurate scientific calculations involving problems where the distance above the earth is relatively short. In such work, however, the more precise formula $d = 16.1t^2$ is used instead of $d = 16t^2$.

Exercises 4-2

(Class Discussion)

The following exercises give some situations which lead to geometric models. Try to describe these models and, if possible, indicate what assumptions about the real world you have made to get your model.

1. On a shelf in a market stand two cans of beans. The first is twice as tall as the second, but the second has a radius twice that of the first. If the second can costs twice as much as the first, which is the better buy?

2. Napoleon's forces, marching into enemy territory, came upon a river whose width they did not know. Napoleon demanded of his officers the width of the river. A young officer immediately stood erect on the bank and pulled the visor of his cap down over his eyes until his line of vision was on the edge of the opposite shore. He then turned and sighted along the shore and noted the point where his visor rested. He then paced off this distance along the shore. Why was this distance that he paced off an approximation of the width of the river?

3. In book-binding a large sheet is usually printed so that after folding and cutting, the pages appear in proper position.

   (a) Suppose that a large rectangular sheet is to be folded, left-right, and bottom-top, forming an 8-page section. Determine, before folding, the proper numbering of the pages.

   (b) How should the position of the print on each page be located so that when folded it's not upside down?

4. How many square inches of skin do you have?

4-3. How Do You Pack Your Marbles?

Recently a company marketed a salt with grains that were diamond shaped crystals instead of just ordinary cubes. The claim, which appears to be true, is that the grains, shaped like diamonds, do not "bounce" off your food like ordinary salt does. However, it turns out that it takes a larger box to *pack* the same weight of the diamond shaped salt compared to the space needed for the same weight of ordinary salt. This means that cooks using this salt must increase the amount used by about $\frac{1}{3}$ in order to have the same amount of salt that they would have used if they had used ordinary salt. (Have you ever tried to measure out $\frac{1}{3}$ of $\frac{1}{2}$ of a teaspoon of salt?)

The situation described above introduces a problem which has some interesting applications. In the design of insulating materials one is interested in having air space in the form of small "pockets" of air which are not large enough to permit circulation. One way to simplify such problems is to create a *mathematical model*. That is, to consider the packing of small spheres, like marbles, between two layers of hard material. Sometimes the surface area of the spheres must be taken into account, as well as the physical properties of the materials themselves. Similar problems occur in the design and testing of plastics.

Suppose that you have a large number of perfectly spherical marbles which you want to pack into a very large barrel. How should you pack the marbles so that you get in as many as possible?

The following exercises will develop an answer to this question and illustrate some of the important procedures involved in creating mathematical models.

Exercises 4-3

(Class Discussion)

1. Our first *simplifying assumption* was contained in the statement that we had a "very large barrel". So now we are really just asking, "How do you pack the marbles so that you have the greatest number marbles per cubic foot, in, for example, a box?" We still need to do more in order to bring the problem down to a level where there's some hope of solving it. We need to make some additional simplifying assumptions.

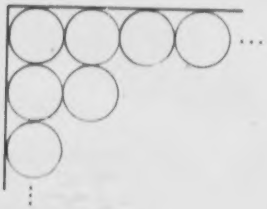
(a) Figures in three dimensional space are sometimes complicated to think about. What figure, in the plane, is related to a box?

(b) What figure, in the plane, is related to the spherical marbles?

(c) State a problem similar to the "marble-barrel" problem using the figures suggested in parts (a) and (b) above.

2. Suppose we use pennies as identical circular disks and see how many we can place side by side, without overlapping in a given plane region.

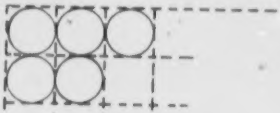
(a) If you arrange the pennies in a square in the following way, how many do you think you could pack in the square?



Assume that the diameter (the distance across) of the penny is the unit of distance, and that the square is 8 units on a side.

(b) This method really amounts to thinking of each circle as inscribed in a square one unit on a side, like this,

and then fitting squares together so that their interiors fill out the square region. (They actually cover about 78.5% of the region, leaving about 21.5% uncovered.)

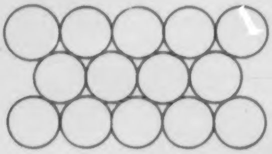


How many pennies does each penny in the middle of the arrangement touch?

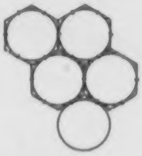
(c) Can you think of another way of fitting the pennies together so that the "inner" pennies will touch more than four pennies? Draw a picture illustrating your method. How many other pennies do the "inner" pennies touch?

(d) How many pennies can you put into your 8 unit square with your new "packing" method?

Probably the following arrangement occurred to you as a good possibility.



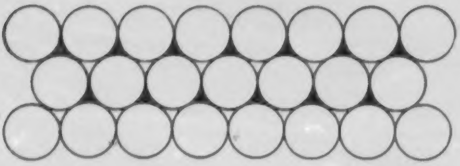
It is helpful to think of each disk as being inscribed in a regular hexagon and the hexagons fitted together as shown by the dotted lines in the figure to the right. Hexagons will fit together to cover a plane region without overlapping. In this arrangement some 90.7% of the plane region is covered by the disks, which is far better than our previous arrangement.



Can you think of other arrangements of the pennies which might be better than this? Actually the arrangement above can be shown to be the best one, though we will not try to give a proof of this fact.

Now let us return to the problem of packing the marbles in the barrel. Do you have a guess about the best way to pack them? Experiment with some marbles before you go on and see if you can make a guess as to the most efficient packing.

One possible procedure would be to start by putting a layer of marbles with their centers all in a plane parallel to the bottom--that is, a layer on the bottom of the box or barrel or whatever we are filling. From the top, they will look just like a covering of a plane region with circles, and in the light of our discussion about circles it seems that we should arrange our spheres in the same way, as shown below.



Now it seems plausible that we should try to make a second layer of marbles. Do you think it would be a good idea to place this layer of marbles such that each marble is directly above one in the first layer? No, it appears that we would get a better packing by trying to place the marbles of the second layer over the "pockets" or "holes" in the first layer. Actually there is not room to put a marble above each hole, but we can place a layer above half of the holes, say those shaded in the figure above. Then a third layer can be placed on the second, covering half of the "holes" in the second layer. This can be done in two ways, depending on which set of "holes" we choose to fill. The spheres of the third layer may be exactly above those of the first or may be above the unshaded "holes" in the first layer.

This seems to be a reasonable guess about the best possible packing. It can be shown that for this method of packing the ratio of the volume of the marbles to that of the region is $\frac{\pi}{3\sqrt{2}} \approx 0.7405$, so this packing fills about 74% of the space with spheres. No one knows whether or not this is really the best packing. The best result known so far was obtained by a British mathematician, Rankin, in 1947, who showed that there is no packing in which the spheres fill more than 82.8% of the volume of space.

4-4. Some Other Mathematical Models You Have Known

In the first three chapters you developed several very useful ideas which will help you create good mathematical models of real life situations. The obvious ones are the ideas about points, lines, planes, tables, graphs, functions, algorithms, and flow charts. These ideas, along with your background in arithmetic and measurement, make up a powerful set of tools that you can use to investigate many significant problem situations. As you proceed through your mathematics courses you will continually expand and refine these and other important ideas. This should enable you to think about an even greater variety of practical situations.

The following exercises illustrate the use of some of the mathematical ideas you have developed, in the construction of appropriate models for some different situations.

Exercises 4-4

(Class Discussion)

1. One of the interesting problems facing those who design computers is how to design a computer that will translate a foreign language into English. Let's "tackle" a simpler problem. Suppose you are asked to find a step-by-step procedure (an algorithm) which will translate Roman numerals into ordinary Arabic numerals, (say numerals whose values are less than or equal to 1000), can you write such an algorithm? The following table shows letters used by the Romans to write their numerals.

Arabic numeral	1	5	10	50	100	500	1000
Roman numeral	I	V	X	L	C	D	M

The values of the Roman symbols are added when a symbol representing a larger quantity is placed to the left in the numeral.

MDCLXVI = 1000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666.

DLXI = 500 + 50 + 10 + 1 = 561.

When a symbol representing a smaller value is written to the left of a symbol representing a larger value, the smaller value is subtracted from the larger.

IX = 10 - 1 = 9.

XC = 100 - 10 = 90.

The Romans had restrictions on subtracting.

(1) V, L, and D (symbols representing numbers that start with 5) are never subtracted.

(2) A number may be subtracted only in the following cases:

I can be subtracted from V and X only.

X can be subtracted from L and C only.

C can be subtracted from D and M only.

Addition and subtraction can both be used to write a number. First, any number whose symbol is placed to show subtraction is subtracted from the number to its right; second, the values found by subtraction are added to all other numbers in the numeral. Notice that the table defines a function f : Roman numeral → Arabic numeral.

(a) Start at the left side of the Roman numeral and by proceeding from left-to-right, write out a basic algorithm for the translation process.

(b) Construct a flow chart for your algorithm and see if it will translate MCLIV into ordinary numerals.

(c) Do you think that something like your algorithm would work for translating a foreign language into English?

(d) What are some of the simplifying assumptions you might have to make in order to create such a model of a "translation system"?

2. Almost everyone is aware that any kind of work (even thinking), causes fatigue. You also know that when you get tired, you can rest awhile, recover, then go on. Suppose we want to investigate the effects of fatigue and to find out how rapidly a person recovers from physical exertion. Such studies are obviously important for biologists, physical therapists, physicians, astronauts, and the like. To see how these studies might be conducted, let us perform a simple experiment and construct a mathematical model of the situation.

An Experiment: Work in teams of three, consisting of a subject, a timer and recorder, and a counter.

The only equipment you will need is a watch with a second hand, some paper, and a pencil.

The first person will be the subject.

(1) He will use one hand in the experiment, the left hand.

(2) Sitting in a comfortable position, his arm straight in front of him resting on the desk top, fingers together, palm up, he should open and close his hand as fast as he can. He should be quite sure each time that his fingers touch the desk when the hand is open, and his fingertips touch the palm when closed.

The second person will be the timer and recorder.

(1) He will watch the second hand, start the subject, and call time at the end of 90 seconds.

(2) He will also record the total number counted by the "counter" at the end of each 15-second period.

(3) The "subject" will now be allowed to rest for 30 seconds, then start the exercise for another 90-second period. As before, the count should be recorded at the end of each 15-second period.

(4) The timer should also see that the subject's fingers straighten completely, touch the table top, and then close until the tips touch the palm.

The third person is the counter.

(1) The counter will watch carefully and count the number of times the subject's fingers touch the table top.

(2) It is important that he count quickly, but aloud, so that the timer-recorder can hear and record the count at the end of each 15-second period.

(3) To get the number of times the fingers were opened for each 15-second time period after the first period the recorder should subtract the first total count from the second total count, the second total count from the third, etc.

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The following is a table of sample data. Make a table like this for each member of your team and rotate jobs until everyone has had a chance to be a subject.

Muscle Fatigue

Name of Subject

Time Period	Time in Seconds	Total Count for Left Hand	Count per Time Period for Left Hand	Difference between the count in one period and the count in the next time period
1st	15	38	38	--
2nd	15	72	72 - 38 = 34	38 - 34 = 4
3rd	15	104	104 - 72 = 32	34 - 32 = 2
4th	15	132	132 - 104 = 28	32 - 28 = 4
5th	15	157	157 - 132 = 25	28 - 25 = 3
6th	15	179	179 - 157 = 22	25 - 22 = 3
REST	30	REST	REST	
7th	15	26	26	--
8th	15	50	50 - 26 = 24	26 - 24 = 2
9th	15	72	72 - 50 = 22	24 - 22 = 2
10th	15	92	92 - 72 = 20	22 - 20 = 2
11th	15	111	111 - 92 = 19	20 - 19 = 1
12th	15	129	129 - 111 = 18	19 - 18 = 1

(a) Draw a graph of your data where the input is the number of the time period and the output is the "Count per Time Period".

(b) Does your graph indicate that you got tired? Did you "recover" fully in the 30-second rest period?

(c) Draw a graph of your data where the input is the number of the time period and the output is the difference between the counts in two successive periods.

(d) Can you use your mathematical model to predict what the "count" would have been in the 7th time period if you had not rested? If so, explain how you could do this.

(e) Can you use your mathematical model to predict what the count would have been in the 13th period if you had continued the experiment? If so, explain how you could do this.

4-5. Summary

A mathematical model tries to duplicate some of the actual characteristics of a real life situation. If these characteristics are properly represented in the model, then we can use the model to predict what might happen in different situations. To be successful a model should:

(1) contain as many of the main characteristics of the real life situation as possible;

(2) the characteristics of the real life situation that are included in the model should behave in the model like they do in the real situation; and

(3) the model should be simple enough so that the mathematical problems that are suggested by the model can be solved.

It should be clear that a mathematical model is never a perfect representation of a real life situation. Usually many "simplifying assumptions" have been made before the mathematical model is finally constructed. The answers found by solving the mathematical problems are not the answers to the real problem situation, but just predictions of what will be seen when the real situation is observed.

Chapter 4

APPLICATIONS AND MATHEMATICAL MODELS

Mathematics cannot deal directly with physical objects. Mathematics can only talk about idealized objects such as points, lines, numbers, and functions. These objects are abstract creations of the mind and have no existence in the real world. In order to use mathematics to solve problems about real life objects, we must first create a "Mathematical Model" in which the real life objects are represented as mathematical objects.

In different types of problems, the same physical objects may be represented mathematically in different ways. For example, when we draw geometric figures on a sheet of paper, we think of the sheet of paper as representing a plane. However, when we have a problem involving the volume of a book, we think of the sheet of paper as a box shaped solid with one dimension very small compared with the other two. We would say that we have chosen different mathematical models appropriate to the different types of problem.

All this is obvious enough when one thinks about it, but the working out of it in practice is fairly subtle and has seldom, if ever, been made explicit for secondary school students. The purpose of this chapter is to make a very small beginning toward making explicit the uses of mathematical models in problem solving.

The recommendation that school mathematics curricula do a better job in making clear the relationships between mathematics and its uses has been a feature of every suggestion for "reform" of mathematics education since at least 1900. The matter seems even more urgent today because of the extraordinary expansion of the use of so-called "mathematical models" in a wide variety of fields that have pretty much done without mathematics before now. For example a list of books recently circulated for college mathematics teachers intended to "give those interested a reasonably clear idea of the nature of mathematical models and the techniques being used in these applied fields" had 63 titles under a Biology heading, 41 titles under Economics, and 63 titles under Social Sciences. In addition, the latest literature in many fields includes "mathematical models" either in the title or in the development of many reports.

The more or less standard rhetoric which describes how mathematics is applied is indicated by the following brief summary of an excellent article printed nearly 20 years ago in which R. S. Burrington explained applied mathematics in terms of "models":

> When considering problems that are concerned with applying mathematics to situations in the real world, one is often confronted with the issues in a complex environment full of distraction. It remains to develop a well organized structure so that the essentials of the problem can be viewed with less confusion. The delicacy of such a task lies in the following:
>
> (1) Removal from the original setting of only the barest features of the problem. This requires due examination of the original setting to gain direction in determining that which is fundamental. The result of such an effort is a simplified, idealized concrete or physical model of the original problem.
>
> (2) This idealized model is to be made the subject of mathematical investigation by direct translation to mathematical terms, i.e., an isomorphism. Essentially this translation is a mathematical model of the idealized model of the original problem.
>
> (3) Through manipulative computation a solution is obtained for the mathematical model. In this stage there is no reference to the original setting or to the idealized concrete model.
>
> (4) The solution is interpreted in terms of the idealized model.
>
> (5) Finally, the solution is interpreted in terms of the original problem.
>
> The validity of the results must be verified and depends upon the extent to which the models include all the known pertinent facts.[1]

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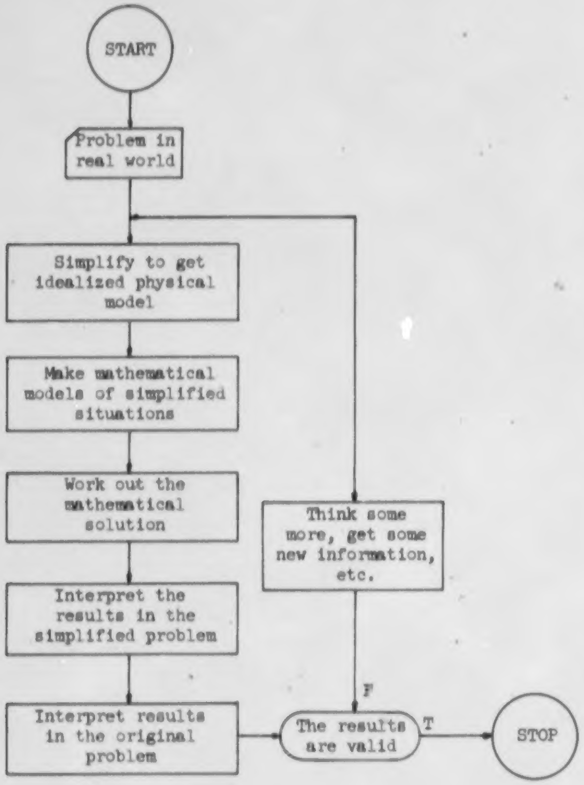
[1] R. S. Burrington, "On the Nature of Applied Mathematics, "*American Mathematical Monthly* (April 1949), 56: 221-242. This has been reprinted in: *SMSG Studies in Mathematics*, Volume XVI, *Some Uses of Mathematics: A Source Book for Students and Teachers*. If you become interested, you may want to read the entire article and perhaps the article "Mathematics and Social Policy" by P. J. Davis, also in the SMSG volume.

After a decent interval of time and after several more sessions devoted to examining how mathematics is applied, a "flow chart" of the process described by Burrington might be developed by teacher and class together, or simply displayed by the teacher on a poster or bulletin board. For example:


START
Problem in real world
Simplify to get idealized physical model
Make mathematical models of simplified situations
Work out the mathematical solution
Interpret the results in the simplified problem
Interpret results in the original problem
Think some more, get some new information, etc.
F
The results are valid
T
STOP

In simple, everyday, applications one does not, of course, go explicitly through all of the steps indicated in the flow chart.

Suggested Time Schedule

Section 4-1.	Introduction	$\frac{1}{2}$ class period
Section 4-2.	Situations Leading to Geometric Models	1 class period
Section 4-3.	How Do You Pack Your Marbles?	1 class period
Section 4-4.	Some Other Mathematical Models You Have Known	1 class period
Section 4-5.	Summary	$\frac{1}{2}$ class period

ALL EXERCISE SETS IN THIS CHAPTER ARE DESIGNED TO BE USED IN DIRECTED CLASS DISCUSSION SITUATIONS.

4-1. Introduction

In this section we focus upon the process of going from the mathematical idea to the real situation and from the real situation to the mathematical idea.

The section should be completed during one class period. The student is first given a brief introduction to what is involved in using mathematics to describe things in the real world, followed by two sets of Class Discussion exercises. The first of these specifies certain mathematical entities, like $3 + 11$, and asks the students to invent situations for which the mathematical entities would be a model. The second set of exercises turns things around and starts with problems and asks the students to think about what mathematical entities would be appropriate as models in attacking these problems.

The exercises should be done in class discussion with a free, game-like atmosphere prevailing. The teacher has the responsibility for setting up the situations in such a way that an interchange of ideas takes place, and also for judging examples offered by students to make sure they are appropriate to the points at hand.

In handling the discussion, the teacher should keep at least the following things in mind.

(1) Nearly every student should participate. Ask them to write down their own ideas, and then discuss, orally, a few of these

suggestions. If some students are not able to handle the examples as given, then give them specific numerical expressions, e.g., "What is a real situation that leads to $2 + 3$ as a mathematical model?"

(2) Be on the lookout for quite different situations which are suggested for the same numerical model. You should especially watch for situations that illustrate quite different concepts, as is possible in Exercise 3 below.

Exercises 4-1a (Class Discussion)

1. "In six months Mr. Adams earned more than $7000. How much did he earn per month?"

   Note: Of course there are many situations where the question could be, "Who has the most ... ?", "Who is older?", "Who has the best (least) golf score?".

   Some other situations are:

   (1) "The distance an object falls during the first second is 32 feet less than the distance it falls during the second second. During the two seconds it falls less than 48 feet because of air resistance. How far does it fall during the second second?"

   (2) "Teddy is more than 3 years old. How old is Teddy?"

   (3) "A student has test grades of 75 and 82. What must he score on a third test to have an average of 88 or higher?"

2. "The total number of grains of sugar in three cups of sugar." Some other situations are:

   (1) "The total number of fish in all the oceans."

   (2) "The total population of insects in the 50 states."

   Note: It would be appropriate to note that the numbers used in these situations are probably approximations to the actual situation.

3. "Jim has 10 marbles and gives 2 of them to John. How many does Jim have left?" (The model is (10 - 2).)

or

"Jim has 10 marbles and John has 2 marbles. How many more does Jim have than John?" (The model is (10 - 2).)

Note: There are many "take away" and "how many more" situations possible. This is a good time to introduce the idea that different real world situations can lead to essentially the same mathematical model.

4. "If there are 224 boys in Gym first period, how many squads of 8 boys each can be formed?" (The model is $\frac{224}{8}$.)

or

"If there are 224 boys in Gym first period, how many boys would be in each of the 8 classes that were formed?" (The model is $\frac{224}{8}$.)

or

"An award of $224 was to be divided equally among 8 people. How much did each person receive?" (The model is $\frac{224}{8}$.)

Note: Hopefully the students will see as "different":

(a) Given a certain amount, how many are there in each of a certain number of groups?

vs. (b) Given a certain amount, how many groups of a certain size can be formed?

5. "The speed of a car at any given moment." (The model would be, for example, 40 mph.)

"The air pressure in a tire." (The model would be, for example, 32 pounds per square inch.)

Note: There are many other similar measure situations. It is likely that many of these involve rounded-off approximations.

6. "The number of feet in 117 inches." (The model would be $\frac{117}{12}$.)

"The perimeter of a square lot is 1213 yards. What is the length of one side?" (The model is $\frac{1213}{4}$.)

"A recipe that makes five dozen cookies calls for 3 ounces of vanilla. How much vanilla should be used if a person wants to make only one dozen cookies?" (The model would be $\frac{3}{5}$.)

7. "What is the elevation of the lowest point in the U.S.?" (The model is -282 ft.) (This is in Death Valley, California.)

"The temperature at the North Pole during a blizzard."

"The actual amount of money one has if he charges \$150 worth of clothes one week and earns \$125 during the same week."

8. "How can you find the height of a giant redwood tree if it stands on an island on which it is impossible to land?"

"How many board feet of lumber is contained in a log at a lumber mill?"

"How much waste is there if the largest possible circular region is cut from a given square region?" (This problem has obvious industrial implications.)

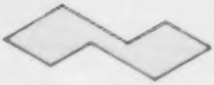
9. "The number of hours a student spends on homework in 5 days." (The model could be $f : t \to 5t$ where t is the number of hours a student spends on his homework per day.)

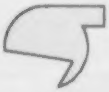
Exercises 4-1b (Class Discussion)

1. The discussion should lead gently to the fact that this is always possible in any convex region like:

etc.,

but that it is not always possible in non-convex regions like the following:





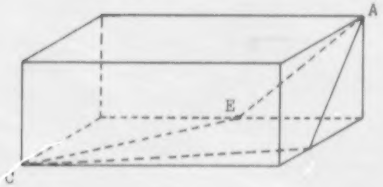
The words convex region are not important at this time. The intuitive realization that there are some geometric regions where it is possible to connect any two points in the region and that the line segment joining those two points will lie entirely within the region is worth some recognition.

2. The main point in this situation is that a mathematical model will be used to evaluate the talents of these two baseball players. It would be difficult to precisely describe these particular skills without a mathematical model since the mathematical model essentially recreates 225 different situations for one man and 183 different situations for the other player. Notice that in this situation the mathematical model has the characteristic of being more permanent than the real life situation.

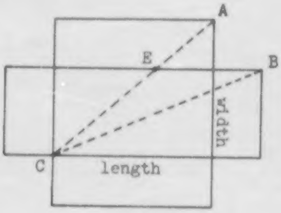
   A comparison of the rational numbers, $\frac{53}{225}$ and $\frac{43}{183}$, usually in decimal form, will help determine who has the best batting average. In this case $\frac{53}{225} = .23555...$ and $\frac{43}{183} \approx .23497$, hence

$$\frac{53}{225} > \frac{43}{183} .$$

3. The model that we would like the student to think of is a rectangular box.



However, to solve the problem easily it is helpful to represent the walls as "flattened" out, that is, in the same plane as the floor.



There are two possible paths represented by segments $\overline{AC}$ and $\overline{BC}$. Usually AC is the shortest distance unless the width of the room is larger than the length.

No specific room dimensions were given since we wanted to avoid getting entangled with the Pythagorean Theorem at this time. We think that the students will agree with the intuitive idea that the line segment, $\overline{AC}$, will represent the shortest path. In the model the shortest path will be path AEC.

4. The comparison of the rational numbers $\frac{5}{8}$ and $\frac{7}{11}$ helps direct the students' decision-making process in this *socially critical* situation. If your students don't know how to compare numbers written in this form, you might be able to gain some "additional mileage" from the situation by *gently* pointing out that there are some situations where such a comparison is important.

$\frac{7}{11} > \frac{5}{8}$ so the student should choose table B. (The 8 and 11 in the denominator are the total numbers of people.)

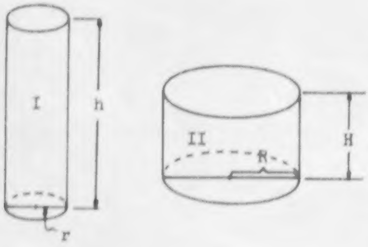
4-2. Situations Leading to Geometric Models

In this section we tried to select some simple problem situations where the need for some geometric model was painfully apparent. Help your students over the "rough spots" in any mathematical calculations. We are trying to develop some awareness, on the part of the students, about how to apply mathematics and we don't want them to get "lost" in some complicated calculations at this time.

It is also recommended that you discuss this section, after the students have read it, prior to doing the exercises. The section points out the simplifying assumptions made in constructing the model of the falling body problem. It then briefly discusses the limiting effect that these simplifications have on the use of the model in interpreting real situations. The student should acquire a feeling that absolute faith in a mathematical model can sometimes be disastrous, but that if he is careful about his assumptions the model can have pretty good predictive qualities.

Exercises 4-2 (Class Discussion)

1.



Help the students with the words "radius" and "height" if they need it.

$$\left.\begin{aligned} V_I &= \pi r^2 h \\ V_{II} &= \pi R^2 H \end{aligned}\right\} \text{ In this situation } h = 2H \text{ and } 2r = R.$$

$$\frac{V_I}{V_{II}} = \frac{\pi r^2 h}{\pi R^2 H}$$

$$= \frac{r^2 \cdot 2H}{4r^2 \cdot H} = \frac{2}{4} = \frac{1}{2}$$

$$\therefore\ 2V_I = V_{II}$$

$$\frac{V_1}{V_2} = \frac{\pi \cdot 5^2 \cdot 12}{\pi \cdot 10^2 \cdot 6} = \frac{25 \cdot 12}{100 \cdot 6} = \frac{1}{2}$$

$$2V_1 = V_2$$

According to the model neither can is the better buy.

Note: We don't want the students to go through the formal analysis given above. If you can lead your students to suggest that you need to compare "how much each can holds" you should be able to convince them of the arithmetic, especially if you use specific numerical values like $r = 5$, $R = 10$, $h = 12$, and $H = 6$, as shown on the previous page.

It's hoped that this problem situation is common enough that it will provoke the students into making a guess, and then to evolve some method to support their guess.

Some simplifying assumptions are:

(1) the top and bottom of the cans are "flush" with the sides;

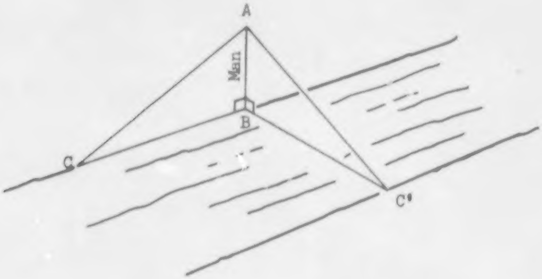
(2) the cans are made of metal with the same thickness;

(3) both cans are "full of beans" (no water); and

(4) the quality of the beans in both cans is the same.

Your class will probably think of many more.

2. The model we're thinking of looks something like this:



Of course students don't know anything about congruent triangles but the process of estimating distances by "sighting" should be familiar. Once the student has been led to represent the first part of the situation by $\triangle ABC'$ it shouldn't be too hard to get him to swing this triangle around until it is in the same position as $\triangle ABC$.

Some simplifying assumptions are:

(1) the man can see along the bank of the river, (no trees);

(2) the bank and river are in the same plane;

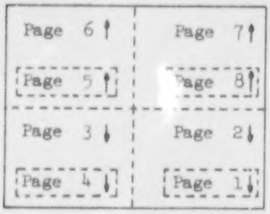
(3) the man can hold his eyes in exactly the same position as he turns;

(4) the angle formed by the man and the ground is a right angle;

and many others.

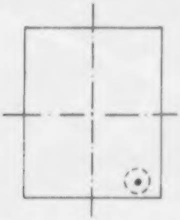
3. The sheet should look like the following:

(DON'T ALLOW THE STUDENTS TO DO ANY FOLDING UNTIL THE END OF THE DISCUSSION!)



The arrows point to the top of each page. The boxes, ⬚, enclose the page numbers that are on the back of the large sheet.

(a) (1) Have the students draw a sketch of one side of the sheet including the "folding lines."



From this sketch the students should observe that four pages are going to be printed on each side of the large sheet.

(2) By looking at some books, if necessary, help the students realize that, starting with the first pair of pages ① and ②, the pairs of pages will be printed on opposite sides of the large sheet.

(3) Now, have the students imagine a point on the "backside" of the lower right-hand region of their sketch, (indicated on the drawing ◌). Then say, "What happens to the point when the first fold is made?" Then ask, "What happens to the point when the second fold is made?" This procedure should help them locate pages ① and ②.

(4) Next, imagine a point on the "frontside" [illegible] the lower left-hand region. Ask, "What page is this point going to be next to after the first fold?" "Does this position change after the second fold?" These questions should help the students locate pages ③ and ④.

(5) The rest of the page numbers can be located in a similar manner.

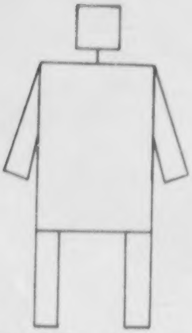
(b) Now have the students put an arrow, ↑, in any one of the two lower regions in their sketch. Ask, "What happens to it after the first fold?" Then, "What happens to it after the second fold?" Hopefully the students should see that the orientation of the print on the bottom half of the sheet should be "upside down."

The students could now take a sheet of paper, number the pages, indicate the top, and then fold it to see if it works. (That is, check out their model with the real situation.)

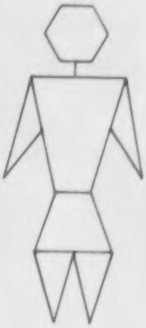
As a brainbuster, some students might like to try a 16 page situation where the folds are L-R, B-T, L-R.

4. Possible models suggested might be:

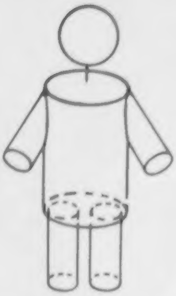
Model 1



Model 2



Model 3



Some of the simplifying assumptions are obvious. We are representing a three-dimensional, irregularly shaped object with planar figures in the first two models and regular solids in the third model. Your students will undoubtedly come up with many other, sometimes bizarre, suggestions for approximating this area.

4-3. How Do You Pack Your Marbles?

The "packing problem" and its applications will most likely present the students with an unfamiliar situation. This is intentional, since our objective is to have the student think about an unfamiliar situation, make some simplifying assumptions and bring the situation down to a level where there is some hope of solving the problem. Then we wish to indicate how the solution of the simplified problem can lead to an adequate solution of the original problem.

It is hoped that actual handling of both the disks and the spheres will develop real geometric insight into the meaning of these packing problems. If another reference is desired the problem is discussed briefly on pp. 148-151 of Rouse Ball, Mathematical Recreations and Essays (revised by Coxeter).

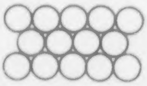
One point should be emphasized. In an actual packing of either disks in a region of a plane or spheres in a region of space, there is almost always waste space around the edges of the region. For example, in Exercise 2(d) below, where pennies are being packed into a rectangular region, every other row will have about half a penny of waste space at one end or the other. This is the reason why the actual results on the part of the area covered by pennies is less than the theoretically computed result. As the region gets larger and larger, this boundary waste is a smaller and smaller fraction of the whole region and the actual results get closer and closer to the result found by using a mathematical model.

Please resist the temptation to teach all about Areas of Squares and Circles, Hexagons, and the like. Use these ideas informally and provide a "helping hand" in the discussions when the mathematical skills of the students appear to be inadequate.

Exercises 4-3 (Class Discussion)

1. (a) Square or rectangular regions (just square or rectangle is O.K.).
   (b) Circular region or circular disk. (Circle)
   (c) "How many circular disks can you pack into a very large rectangular region so that you get in as many as possible?"
2. (a) Using this arrangement, 64 pennies. (If you have an overhead projector and 64 pennies this could be easily demonstrated.)
   (b) Four
   (c) This is the arrangement we're looking for.

The inner pennies touch six other pennies.



   (d) 68. The pennies cover about 83.4% of the area of the square.

If your pupils seem particularly interested in such geometric problems, you may wish to mention the following problem which is unsolved, but not too hard to understand.

How should n points be placed on the surface of a sphere of radius 1, so that the smallest of the distances between pairs of points is as large as possible? You can visualize this problem by imagining that you have n ants on the surface of the sphere who hate each other enthusiastically and want to stay as far apart as possible. The question is how they would arrange themselves on the sphere. For a few of the simpler cases the solution is easy to see. For example, when $n = 2$ the two points would be placed at opposite points of a diameter. The problem was solved in 1951, by Schmitte and Van der Waerden for the cases, $n = 5, 6, 7, 8, 9, 12$. But even for $n = 10$ the solution is not known.

4-4. Some Other Mathematical Models You Have Known

In this section we present two situations which will use several of the mathematical ideas developed in previous chapters to construct suitable mathematical models. Since these situations are designed to use recently acquired skills and understandings, the students should be able to participate more fully in the development process.

Exercises 4-4 (Class Discussion)

1. The key part of the algorithm is that as you read the Roman numerals from left to right you add their values unless the value of the "left" numeral is less than the next numeral to the right. For example, XIV:

	Sum
(1) The first numeral on the left is X.	
(2) The next numeral is I and the value of X is greater than the value of I so we write down 10, the value of X, in our Sum.	10
(3) Now we consider I. The value of I is less than the value of V, the next numeral, so we subtract the value of I from our sum.	$10 - 1 = 9$
(4) Now we consider V. There are no more numerals so we add the value of V to our sum.	$9 + 5 = \boxed{14}$

Remember that in the development of algorithms and/or flow charts you usually start somewhere in the middle with the key idea and work your way outward to the start and stop stage. The following is an example of a basic algorithm that you might develop with your class. Remember that the Roman numeration system is basically additive, reading from left to right.

(a) Basic Algorithm for Translating Roman Numerals to Arabic Numerals

(1) Write down the value of the Roman numeral on the left.

(2) Write down the value of the next Roman numeral to the right.

(3) Compare the values of these numerals.

(4) If the value of the numeral on the left is greater than the value of the next numeral, then add the value of the "left" numeral to the sum. (The sum starts out with a value of 0.)

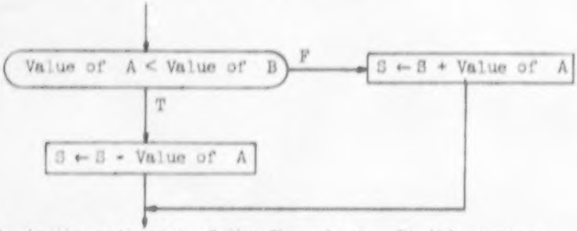
(5) If the value of the numeral on the left is less than the value of the next numeral, then subtract the value of the "left" numeral from the sum.

(6) Repeat steps 1 through 5, shifting over to the left one numeral at a time until all numerals have been considered in step 1.

Note: The key parts of the algorithm are steps (3), (4), and (5).

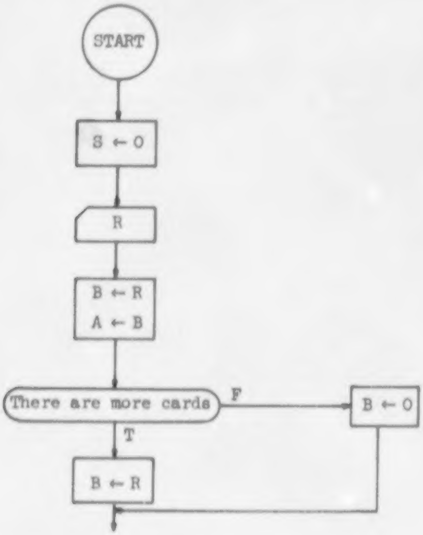
(b) The development of the flow chart should occur in stages with the refinements being added step by step. Some possible stages in the development of the flow chart are shown below.

(1)



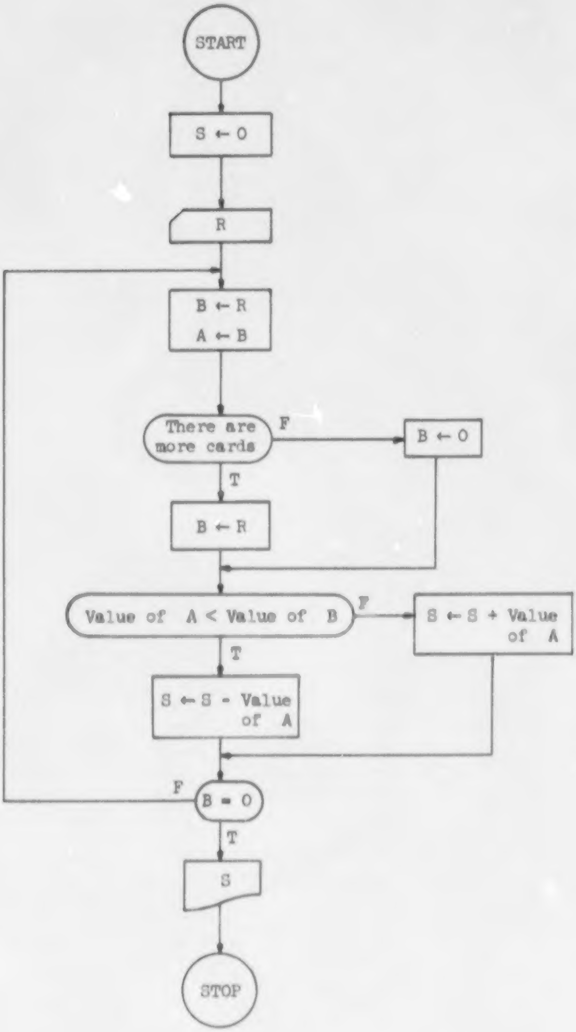
This is the main part of the flow chart. It illustrates the pairwise comparison of the adjacent numerals A and B, and then the addition or subtraction of the value of the numeral A to the sum S.

(2) Now that we have a "middle", let's see if we can devise a beginning. Our next major problem is to simulate the numeral-by-numeral movement from left to right. First we assign the sum S the initial value of zero (box 1). Next we input the Roman numeral (box 2), one numeral at a time, reading from left to right. This input is designated by R. (Remember that once an input is read and assigned that card is destroyed and the next input card comes up to be read.)



(3) By using the "dummy" variables A and B we can now model the inspection of pairs of numerals as we move along the Roman numeral. In box 3, we assign to B the first left-hand Roman numeral, and then we immediately assign B to A. In box 4 we ask, essentially, if we have come to the end of the numeral. If we have read the last numeral, then we assign 0 to B, and this will be our signal to stop the machine.

(4) The following is a complete flow chart for translating from Roman numerals to Arabic numerals.


START
S ← 0
R
B ← R
A ← B
There are more cards
F
B ← 0
T
B ← R
Value of A < Value of B
F
S ← S + Value of A
T
S ← S - Value of A
F
B = 0
T
S
STOP

2. A watch or clock with a second hand is necessary for timing purposes.

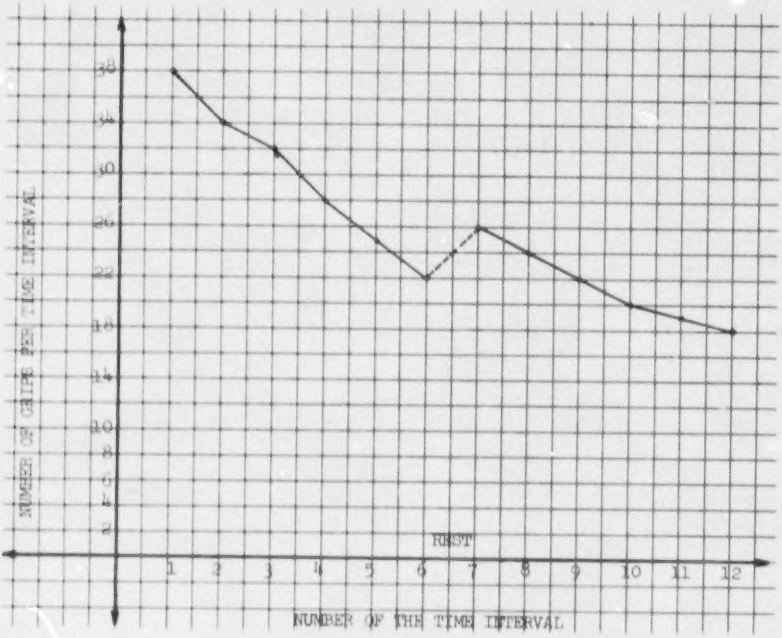
It is most important that the students be cautioned to follow instructions exactly. Touching the table and the palm of the hand with the finger tips every time is essential. Some good-natured policing will undoubtedly be necessary. (It was necessary even among adults!)

If a student shows an increase in number per 30 seconds, he is not doing it properly.

Be sure the instructions are thoroughly preread and that the students understand the activity before beginning.

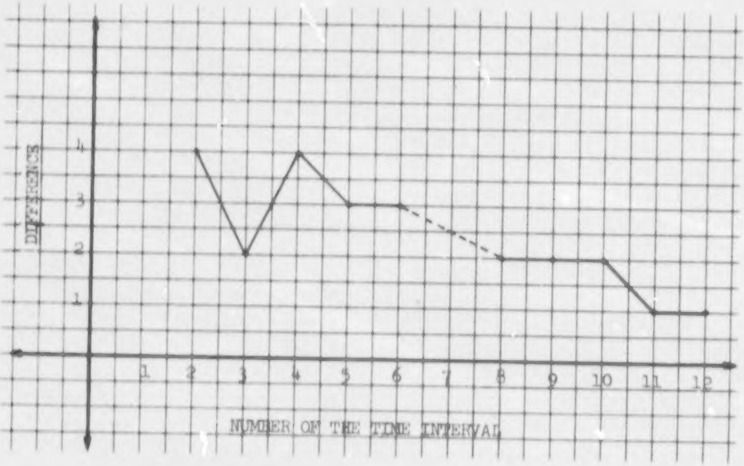
The rest period should not allow complete "recovery", so the seventh time period should read higher than the sixth but not as high as the first.

(a) (Sample graph of data given in text.)



(b) The graph constructed from the sample data shows a definite fatigue in muscles and a slight recovery with rest. The students' graphs will vary considerably.

(c)



(d) The graph of the sample data in part (c) indicates that the decrease had become constant, so a good estimate here for the 7th interval with no rest, would be 19 grips.

(e) A good estimate from the graph of the sample data for the 12th interval is 17 grips.

Extrapolation is the estimation of the value of a function beyond an interval from values of the function within the interval. Here we are just making an educated guess of the next value based on previous information. One must be careful because if the data is limited, it is easy to arrive at a wrong conclusion.

Suggested Test Items

It is strongly recommended that no test be given on this chapter. An informal discussion of the ideas presented in the following article might serve as a satisfactory culminating activity.

Excerpts from
"Philosophical Remarks on Model-Making"

by M. L. Juncosa
RAND Corporation

A search for an "understanding of phenomena" has dominated human intellectual activity from the beginning of time. This pursuit of knowledge of structure and of causation can be motivated by desire for comfort, fear of the unknown, satisfaction of curiosity, and so on.

For the scientist--and this includes mathematicians--a strong motivation is the desire for predictability; that is, within certain bounds the structure or model can be interpreted as being "consistent." For the inductive scientist, results of the theory, i.e., predictions, will "agree" with experiments. For the mathematician, contradictory theorems will not result.

To arrive at conclusions, a process of what some people call model building is engaged in. The primitive man invents concepts of supernatural gods with anthropomorphic attributes, such as anger at broken taboos, and enormous powers, such as the power to cause awe-inspiring earthquakes, eclipses, and so on. The scientist observes physical, economic, sociological, biological, or psychological phenomena: He invents idealizations of the phenomena at hand according to some laws which may exist from previously studied "similar" situations or which he constructs for the purpose. These contain what he feels is the "essence" of the observations. As what he calls consequences of these laws, he makes certain statements or predictions, asserting that he has now an explanation, a theory, or more modestly, a model (not necessarily unique and which may or may not be mathematical) for the phenomenon. The predictions are checked out to see how valid they are and, if necessary, the model is changed.

The mathematician does not necessarily deal with real world phenomena directly but does construct many conceptual models of other concepts or

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theories that he is in the process of exploring. He frequently finds that in the model or image he may have greater insight or may be able to use language which is not quite available for the original. This enables an "end-run" in the proofs of goal theorems or suggests new goal theorems and techniques for the original. It is not uncommon that he makes physical models and pictures as models for his theory for greater elucidation and inspiration.

It is essential to recognize the universality and variety in the philosophy of modeling, regardless of what it is called. Not only does everyone do modeling at some intellectual level, but frequently transitions from one world to another and back again are made. An engineer may make a mechanical model (here called analog) of springs, weights, and dash-pots for an electrical circuit of resistances, capacitances, and inductances, or vice versa. If he has a mathematical model as well, he may not even make, i.e., physically construct, the mechanical or the electrical analog but will rather solve the pertinent equations and then interpret the results in either the original situation or the analog, using the language of the one most familiar to him because "it is easier (for him) to see it that way."

Continuing with the variety of instances of the modeling activity, we have:

1. the process of going from the real world to the real world, cited above (construction of analog computer, slide rules, etc.);

2. the process of going from the real to the conceptual or mathematical, and then back to the real (mathematical physics, mathematical biology, mathematical economics, operations research, applied mathematics in general, the process being admirably described in Burrington's article, "On the Nature of Applied Mathematics"[2]);

3. the process of going from the conceptual to the real and back again (construction of Venn diagrams and switching circuits for set theoretic and Boolean operations, construction of finite group multiplication tables, construction of rings, trees, graphs, knots, cross caps, Klein bottles for certain topological objects);

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[2] Burrington, _op_. _cit_.

4. the process of going from a conceptual to another conceptual without passing to the real and back again (identification between real numbers and points on a line, language structures as trees or graphs);

5. and even processes of going from lower to higher conceptual levels and back, as well as vice versa.

Some model-making also goes from deterministic conceptions to probability conceptions and back again.

We will restrict our attention here to the philosophy of modeling as applied to real world problems, as indicated in number 2 above. Since the term "mathematical models" is used very extensively in biology, economics, management psychology, operations research, control applications, chemistry, statistical mechanics, etc., we accept this usage. At the same time, we recognize that modeling is a broader intellectual concept and caution strongly against a monopoly on a narrow use of the term that might cause polarization of attitudes among people who should recognize the universality of the process. Such synonyms should always be pointed out with their slightly different shades of connotation indicated.

Another observation on models is the strong essence of approximation present, particularly in real world problems involving either continuous variables or large numbers of variables (e.g., gases, populations, traffic) in some problems. We have simple examples of this in the representation (model) of a flat sheet of paper as a rectangle for most purposes but as a rectangular parallelepiped (!) when one is interested in estimating the volume of a book; the habitable world in antiquity or much smaller localities today as a flat segment of a plane (ignoring the local mountains, valleys, rivers) but the habitable world as a sphere today (or happily for Eratosthenes seeking an estimate of the size of the earth), or the earth as oblate spheroid for satellite work because of the precision required in orbit computation; the circulatory system as a pump; a gas (collection of molecules) as a fluid; etc. These approximations frequently are made to enable the recognition of mass behavior or macroscopic behavior; at other times they are made to make a problem either mathematically tractable or computationally feasible. An essential factor in a good model in this class of situations is that of stability; small deviations in the original should result in small variations in the predicted result.

A poor model in these situations is one with enormous variations in the results for small deviations in inputs. (We should observe that occasionally it is because of the nature of some startling variation in the predictions by inadequate models that original discoveries are made by entirely new formulations.)

In another class of models the essence of approximations does not figure strongly or even at all. In these it is structure that is important: Do the variables in the problem figure linearly or not? Can an algebraic group structure be assumed in the model for the phenomenon or not? Is the model for the world Euclidean (parallel postulate, etc.) or not? Problems for which tree-like or graph-like models are constructed have this flavor. It is important to recognize that the notion of stability seems to be irrelevant here. The familiar problem of the three houses desiring three utilities without overlapping connections from the mains to the homes, modeled as an attempt to construct a certain (impossible) graph of six vertices, is again structural. Stability and approximation considerations are irrelevant, there being no "neighboring" problem. (The "solution" is "possible" as soon as the number of homes is reduced to two.)

Chapter 10

PROBABILITY

10-1. Introduction

A story is told about a lady who bought some flash bulbs for her husband who is an enthusiastic amateur photographer. When the husband complained that none of the bulbs were any good, the lady said, "That's funny, I tested every one and they all worked!"

Of course this is not a true story, but it does point out an interesting thought: "How can a manufacturer be reasonably sure that a shipment contains very few defective flash bulbs when he knows that testing a bulb will ruin it?" Answers to such problems may be approached with the help of probability theory. Here are a few other examples in which the subject of probability exerts considerable influence.

(a) Mr. Ames is running for mayor of Springview, a town which has 28,000 registered voters. The town newspaper conducts a straw poll covering 200 voters taken at random. If 105 of these voters say that they intend to vote for Mr. Ames, what is the probability that he will be elected?

(b) Mr. Franklin is now 30 years old. What is the probability that he will live to be 65 years old? How does a life insurance company make use of this type of information?

Probability is a mathematical topic about which many people have intuitive ideas. It turns out, however, that one's intuition cannot always be trusted. Consequently it is important to develop a firm foundation for the basic notions.

10-2. Uncertainty

Some events are certain. If you go swimming, you are certain to get wet. If you select a boy from your class, you are certain to find that he is more than 5 years old.

Some events are not certain. We use words like "probable", "likely", "unlikely", in talking about them. For example, when a weatherman makes a forecast "Rain", he actually means, "It will probably rain." Similarly, you may predict that, "The Cardinals will win the pennant", but what you mean is, "I think that it is likely that the Cardinals will win the pennant."

We often make decisions about what to do in situations where we cannot be certain of what will happen. Very often these decisions have to be made by "weighing the pros and cons" and finally choosing one of two or more alternatives. The phrase "weighing the pros and cons" is used at this point for a special reason. Ordinarily when we weigh something we measure it--we assign a numerical value to a characteristic of it that we call weight. When we "weigh the pros and cons" we are trying in our minds to give a numerical measure to the circumstances that are "for" one alternative and compare it with a numerical measure of those "against" the alternative. If we can assign numerical values to the pros and cons, we feel happier about our decision.

We are in a supermarket, have collected our groceries in a basket, and push the basket toward the cash registers. Which line do we pick? We try to make a numerical decision--we count people, estimate the number of packages in their baskets, and then choose a line.

Here is another mathematical decision in the supermarket. The manager watching over the store sees the lines at the cash registers getting longer and longer. A voice over the intercom says, "Clerk A to Gate 7, Clerk B to Gate 8." This manager may know from experience how long the lines should be before another man is sent to the cash registers. Notice that he counts the customers. His decision is based on numbers.

Here are several pairs of statements. Which statement of each pair tells more?

1. (a) I think Bill is a better batter than Tom.

   (b) I think Bill is a better batter than Tom. Bill's batting average this year is .300 and Tom's is .190.

2. (a) I think homeroom 207 will beat homeroom 112 in today's game.

   (b) I think homeroom 207 will beat homeroom 112 in today's game. Homeroom 207 has won five of its seven games while 112 has won three out of seven.

3. (a) Weather forecast: rain tomorrow.

(b) Weather forecast: 80% chance of rain tomorrow.

4. (a) Dr. A: "Try this remedy for your sunburn. It may help you."

(b) Dr. B: "Try this remedy for your sunburn. It has helped 6 out of 7 patients who have tried it."

You have probably noticed that in each pair of statements the second would be more helpful, because it gives you more definite information. In each case, the additional information involves numerical measures of some sort. You should realize, however, that the numerical information given does not make the conclusion certain. Tom may have played all year with a sore arm, the better team does not always win, it may not rain, and the remedy may not work for you.

In probability we shall study systematic methods of weighing pros and cons. Although we cannot change an uncertain future to a certain one, we can sometimes compare likelihoods of various occurrences.

One of the objectives of this chapter is to learn how to assign appropriate numerical measures to uncertain events. These measures will be called probabilities.

The study of probability has many practical uses. For example, federal and state governments use probability in setting up budget requirements; military experts use it in making decisions on defense tactics; scientists use it in research and study. Engineers use probability in designing and manufacturing reliable machines, planes, and satellites; business firms use it to help make difficult management decisions; it is the main tool of the insurance industry in deciding on premium rates and on size of benefits.

We will use as illustrations several examples of games of chance, employing such familiar objects as coins, dice, and playing cards. The examples have been chosen since they are fairly simple to understand. We all have some intuition about the "chance" of throwing heads when a coin is tossed. The practical situations indicated in the preceding paragraph are too complicated for the present, although we will mention some special problems from these fields of application.

It is of interest that, historically, the mathematical theory of probability arose from the consideration of gambling games.

10-3. Fair and Unfair Games

Suppose that we decide to play a game for two players in which the outcome depends upon chance, not skill. We agree that the game is fair if each player has an equal chance of winning. For the time being, we will assume that you have an intuitive idea of what we mean by "equal chance".

How may one decide whether a game is fair? Sometimes careful thinking about the rules will enable us to decide. Another possibility is to play the game many times, keeping a record of the results. This may give us a "feel" as to whether the particular game is fair. We would expect that if a game is grossly unfair, several trials might indicate that fact. If a game is almost fair, it may take lengthy experimentation to discover that fact. For some of the games described in the following exercises, you may wish to conduct, say, 20 trials to give you a clue about whether they are fair.

Exercises 10-3a

(Class Discussion)

Here are some games to think about. For each game a rule is given to tell whether you or your opponent wins. If neither wins, the game counts as a tie. In each case decide whether the game is fair.

1. These games are played by two players with a die having six faces, numbered 1, 2, 3, 4, 5, 6, respectively. The die is tossed once.

   (a) You win if the face numbered 1 is thrown. Your opponent wins if 3 is thrown.

   (b) You win if an odd number is thrown, and your opponent wins if an even number is thrown.

   (c) You win if 3 is thrown, and he wins if a number greater than 3 is thrown.

2. These games are played by tossing a die with "1" on one face, "2" on two other faces, and "3" on the three remaining faces.

   (a) You win if 3 is thrown. Your opponent wins if 1 is thrown.

   (b) You win if 3 is thrown, and he wins if any number less than 3 is thrown.

   (c) You win if an even number is thrown, and he wins otherwise.

3. These games are played with two ordinary dice, one white and one green. Both dice are thrown together.

   (a) You win if 1 is thrown on each die. Your opponent wins if 5 is thrown on each die.

   (b) You win if there is an even number on the white die, and he wins otherwise.

   (c) You win if 6 shows on the white die, and he wins if 4 shows on the green die.

   (d) You win if 1 is on each die. Your opponent wins if one die has 1 and the other has 2.

   (e) You win if the number on the white die is greater than the number on the green die. Your opponent wins otherwise.

4. These games are played with one ordinary die. The die is thrown two times.

   (a) You win if the number on the second throw is greater than the number on the first throw. Your opponent wins otherwise.

   (b) You win if the number on each throw is even, and he wins if the number on each throw is odd.

Let us summarize. We have used the idea that a game played by two people (or teams) is fair if winning is as likely as losing. "Winning" means that particular events occur; "losing" means that other events occur. You cannot both win and lose.

Did you discover which of the rules described in the class exercises does not really describe a game, because it permits both players to "win"?

Consider throwing an ordinary die. Throwing one specified number is just as likely as throwing any other specified number. For example, the appearance of 5 is as likely as 2. On the other hand, throwing one specified number is less likely than not throwing that number. For example, it is less likely that 5 appears than that 5 does not appear. Throwing an even number and throwing an odd number are equally likely events.

Exercises 10-3b

(Class Discussion)

1. Throw a die and record which face comes up. Have the class repeat the experiment until 100 trials of this experiment have been made. For convenience in counting and reading, record the numbers in blocks of five with five blocks to a row.

   For example, the first row might look like this:

   24131 64235 26563 22522 21355

2. Make a copy of the form below and use it to record the number of times each face appeared in your experiment. This number is called the frequency and the table is called a frequency table.

	Number on a die face					
	1	2	3	4	5	6
Frequency, first row						
Frequency, second row						
Frequency, third row						
Frequency, fourth row						
Total						

(For the sample row given above, the first row of the frequency table would be

3	8	4	2	5	3

.)

3. Use your data again to complete the table below. This time we are interested in "runs" of like numbers. We want to know how many runs of each length occur--the actual number repeated in a given run is of no interest for this question.

   (a) Copy the form and fill in the second column.

Frequency	Like Numbers	Consecutive Numbers
Exactly two		
Exactly three		
More than three		

[For the sample row above the result would be

2	(22 and 55, blocks 4 and 5)
1	(22 2, last two in block 4 and first in block 5)]
0	

(b) For the third column look at your data and count the number of "runs" of exactly two consecutive numbers in increasing order. For example, in the second block of the sample row the third and fourth entries are the run "2 3". Also in the third block is the run "5 6".

(c) On the basis of your record above, do you think that a pair of consecutive numbers is just as likely as a pair of like numbers? How do you feel about triples of each sort?

4. Here is a record of 25 throws of a die.

2 3 3 3 3   3 3 3 2 2   1 2 1 3 3   3 3 2 1 3   2 3 2 3 1

(a) Do you believe that this record could happen with an ordinary die?

(b) Do you think it is likely to happen?

(c) Can you offer any other possible explanations of the record as reported?

One possible explanation you may have offered is based on the game described in Problem 2 of Exercises 10-3a. As a matter of fact, the record is typical of what such a die would yield. You may wish to try it for yourself. Since you undoubtedly do not have a die whose faces are marked 1, 2, 2, 3, 3, 3, respectively, you can adapt one ordinary die as follows: Think of the face with four spots as being a 2, and think of the faces with five or six spots as being 3's. This is the scheme used by the writers to obtain the record presented above.

Here are two more examples of games which may or may not be "fair".

Example 1. Suppose that a friend suggests playing a game that you think is fair. You play it 100 times. You win 45 times and he wins 55 times. You would probably feel, quite rightly, that this is reasonable enough. You should not expect to win exactly 50 times.

Example 2. Now suppose that in 100 plays of a certain game using a die you lose 95 times. Which of the following statements do you think are reasonable conclusions?

Statement A. The game is fair, and you have had a run of bad luck.

Statement B. The game is fair. If you play the game another 100 times, you will win most of them.

Statement C. The game is not fair. The evidence of 95 losses out of 100 plays is convincing.

You should certainly have not accepted Statement B as reasonable. If the game *is* fair, then in the next 100 plays you should expect to win only about 50 times. The die has no tendency to try to counterbalance its past performance.

Statement A is a possible conclusion, but as you study more about probability you will discover that a run of such bad luck in a fair game is extremely rare.

Statement C seems the most reasonable of the three alternatives. The evidence is convincing, but note that the results of 100 trials cannot *prove* that the game is unfair.

Exercises 10-3c

1. An experiment similar to the one in Exercises 10-3b (Class Discussion) of throwing one die 100 times and recording each time which face is up, was performed, with the following results.

43553	53344	14166	53213	46451
54563	41353	35335	65536	64112
43253	62454	53263	33423	21531
24131	64235	26563	22522	21355

(a) Construct a frequency table for these data as in Exercise 2 of Exercises 10-3b.

(b) Is the total frequency of each number about what you would have expected to get in 100 tosses? Which number occurred more frequently than you would have expected? Which occurred less frequently?

2. (a) For the data in Exercise 1 above, construct a table as in Exercise 3 of Exercises 10-3b to record the frequency of "runs" of like numbers and consecutive numbers.

   (b) On the basis of your table, does a pair of successive numbers seem just as likely to occur as a pair of like numbers? What about triples?

3. Suppose we have an ordinary die with six faces 1, 2, 3, 4, 5, 6. For each of the following games decide whether it is fair or unfair.

   (a) On a single toss of the die you win if the number on the face which comes up is prime; you lose if it is not.

   (b) On a single toss you win if the number which comes up is the square of an integer; you lose if it is not.

10-4. Finding Probabilities

In the preceding section we considered the likelihood that an event occurs. In particular, for two events we attempted to decide whether they are equally likely or whether one is more likely than the other. To describe these ideas more satisfactorily, we need numbers. This section will be devoted to finding appropriate numbers to measure likelihoods.

We begin by asking you to do two experiments, written in terms of colored marbles. If you do not have marbles of the colors mentioned, you may use other colors, or even substitute colored disks or slips of paper for the marbles. Just be sure that the objects you use (1) can be distinguished from each other by color or in some other way, but (2) in all other respects are alike, so that the experiments will not be biased.

Exercises 10-4a

(Class Discussion)

1. Put three marbles, one red, one green, one yellow, all the same size, into a box or a jar that you cannot see through, and mix them thoroughly. Without looking, reach in and draw one marble. Record its color, then return the marble to the container. Mix all the marbles and draw again. Repeat drawing a marble and recording the color until you have drawn 12 times in all.

   (a) Approximately how many times did you expect to draw each color?

   (b) How many times did each color actually appear?

2. Again put three marbles, but this time two white and one blue, into the container and repeat the same process as before: mix, draw a marble, record its color, return the marble. Repeat until you have made 12 trials.

   (a) How many times did each color appear?

   (b) Did you expect the number of whites to be about the same as the number of blues?

In the first experiment above there are three possible "outcomes". Each outcome can be identified by the color of the marble drawn: red, green, or yellow. From the nature of the experiment, it seems that these three outcomes are equally likely. Since we wish to use numbers to express such ideas, we will assign to each outcome a non-negative number. This number we call the probability of the outcome. We shall use the symbol

P (red)

to represent the probability that the marble drawn is red. Notice that P (red) is the symbol for a number.

In a similar manner, P (green) represents the probability that green is the color of the marble obtained on a draw. Can you tell what is meant by the symbol P (yellow)?

P (red), P (green), P (yellow) are numbers. So far we have not said what these numbers are. However, if they are to fit with our ideas about what is likely, we can see that not just any numbers will do. We want the probability of an outcome to be a measure of its likelihood, just as the weight of a block is a measure of its heaviness. It seems sensible, then, to require that:

Equally likely outcomes have equal probabilities.

On the basis of this agreement, the numbers P (red), P (green), P (yellow) should all be the same.

If all three of the marbles were red, we would be certain that if one marble were drawn the color would be red. Suppose that we agree to assign the number 1 as a measure of the probability of drawing a red marble from a container holding 3 marbles, all red. What number, then, do you think would be appropriate to represent the probability of drawing a red marble from a container in which $\frac{1}{3}$ of the marbles are red?

Since the drawing of a red marble is 1 of 3 equally likely outcomes, it seems reasonable to expect that in many trials of the experiment, the color of the marble drawn will be red about $\frac{1}{3}$ of the time. This number $\frac{1}{3}$ is a measure of how likely we believe the outcome red is, and gives us a clue about how to choose the probability for red. The same type of discussion applies to each of the other colors. Thus we assign to each of the three outcomes the probability $\frac{1}{3}$. In symbols, we have:

$$\begin{aligned} P\,(\text{red}) &= \frac{1}{3} \\ P\,(\text{green}) &= \frac{1}{3} \\ P\,(\text{yellow}) &= \frac{1}{3}. \end{aligned}$$

As another example, consider a spinner with half its area red and half white, like the one shown here.

An experiment consists of twirling the pointer with a good push. By the same reasoning as in the case of the marbles, in this experiment we have:

$$P\,(\text{red}) = \frac{1}{2} \quad\text{and}\quad P\,(\text{white}) = \frac{1}{2}.$$

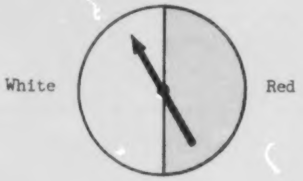

Figure 10-1

We are assuming that the pointer does not land exactly on the boundary between two regions. Indeed, we shall adopt this understanding in all of our examples involving spinners. (This agreement represents an ideal situation.

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Sometimes the spinners in children's games have wide marks for the boundaries, and the stopping of the pointer on such a mark is not unusual. If this happens when you are actually performing an experiment, discard the trial and spin again.)

Exercises 10-4b

1. Consider the experiment of throwing one time an honest die (a cube) in a fair manner.

   (a) How many faces does the die have?

   (b) If an outcome is represented by the number of spots on the face that appears on top as a result of the throw, how many possible outcomes are there?

   (c) Are all of the outcomes equally likely?

   (d) In many repetitions of the experiment, about what fraction of the time would we expect each of the six outcomes to occur?

   (e) To each outcome we assign the fraction described above as the probability. Hence we write:

   P(1) = ______ P(4) = ______

   P(2) = ______ P(5) = ______

   P(3) = ______ P(6) = ______

2. One of the simplest experiments in probability consists of tossing a coin. Suppose that we have an "honest" coin--one that is not weighted in any way. We toss it in a fair manner and let it fall freely. When it comes to rest, it shows either heads or tails.

   (a) For a single toss of a coin, how many possible outcomes are there? Name them.

   (b) Assuming that an "honest" coin is tossed in a fair manner, what is the probability of its showing a head?

   (c) Complete these statements for the coin tossing:

   P (heads) = ______

   P (tails) = ______

   (d) Suppose that you repeatedly perform the experiment of tossing an honest coin, and that on each of the first five tosses the coin shows heads. What is the probability that on the next toss the coin will show tails?

3. (a) Suppose that an honest die has been thrown 20 times without yielding the outcome 5 at all. What is the probability that on the next throw it will show 5?

   (b) Suppose that the spinner shown in Figure 10-1 stops on red in 10 consecutive spins. What is the probability that it will stop on white on the next trial?

In the second experiment with marbles you put two white marbles and a blue one into the container and drew out one without peeking. The likelihood of drawing any one of the three marbles is the same as the likelihood of picking any other. Thus we can again think of 3 equally likely possible outcomes. One of the possible outcomes is choosing the blue marble, so we can write

$$P\ (\text{blue}) = \frac{1}{3}\ .$$

What number might be assigned to P (white)? When you did the experiment you found that the color of the marble drawn was more likely to be white than blue. In fact, you probably found that white occurred about twice as often as blue. Thus it would seem that P (white) should be a number greater than $\frac{1}{3}$, perhaps about twice as large as $\frac{1}{3}$.

Continuing our analysis, we observe that the outcome white occurs whenever either of the two white marbles is picked. The drawing of one specific white marble is one of 3 equally likely outcomes, so its probability should be $\frac{1}{3}$. The same remark applies to the other white marble. It seem reasonable, then, that the probability of drawing one or the other should be $\frac{1}{3} + \frac{1}{3}$, or $\frac{2}{3}$. Thus

$$P\ (\text{white}) = \frac{2}{3}\ .$$

Suppose now that instead of putting two white and one blue marble in the container, we put in three white marbles. If you draw once from the container, what is the probability that the marble you draw is white?

P (white) = ______.

If you throw a single die, what is the probability that the face which comes up shows a number of spots less than 7?

In these last two situations we considered events that were certain to occur. In each case, we found the probability to be 1. In general, we agree that:

If an event is certain to occur, then its probability is 1.

This, then, gives us the unit of measure we choose as a standard of comparison in probability.

We think again of the container with three marbles, all white, and ask: What is the probability of drawing a black marble? This, of course, is not a possible outcome, but we do wish to assign a probability. Each time you repeated the experiment, you would obtain a black marble none of the time, hence we take the probability to be the number 0.

In general, we agree:

An event which cannot occur has probability 0.

Exercises 10-4c

1. Consider a single toss of an honest die.

   (a) Which of the possible outcomes are even numbers?

   (b) What fraction of the 6 possible outcomes are even numbers?

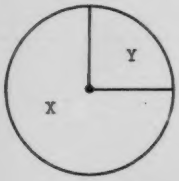
   (c) What number represents P(even)?

   (d) What is P(odd)?

   These probabilities represent our belief that an odd number and an even number are equally likely events.

   (e) Suppose that s is the number of spots shown when the die is thrown. For what values of s is the sentence $s < 3$ true?

2. The spinner shown here is colored white throughout. The region X has an area 3 times the area of region Y. (For simplicity, the needle has been omitted from the drawing.)



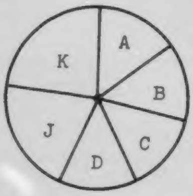
   (a) It is certain that the spinner will land on white, hence P(white) = ______.

(b) It is reasonable to expect that $P(X) =$ _____ $\cdot P(Y)$.

(c) Since region Y has one-quarter the area of the spinner, $P(Y) =$ _____, $P(X) =$ _____.

(d) $P(X) + P(Y) =$ _____, which confirms the feeling that $P(X) + P(Y)$ should be the same as $P(\text{white})$.

3. On this spinner, regions A, B, C, D are all congruent. The area of each is $\frac{1}{7}$ the area of the spinner.



(a) $P(A) = P(B) = P(C) = P(D) =$ _____.

(b) $P(A) + P(B) + P(C) + P(D) + P(J) + P(K) =$ _____.

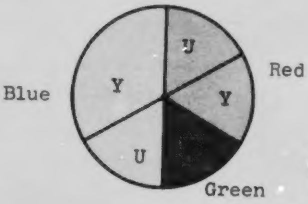
(c) If $P(J) = \frac{1}{5}$, then $P(K) =$ _____.

10-5. Outcomes and Events

In analyzing complicated situations, it is sometimes useful to have some special terms. We have spoken of an experiment consisting of one spin of a spinner. Sometimes an experiment consists of several repetitions of a certain action; in such a case we speak of each performance of the action as a trial.

For an experiment, before we perform it, we can give a set of possible outcomes, or simple events, that can happen. When you perform the experiment in accordance with the rules set up for it, you are certain to get exactly one outcome out of the set of possible outcomes.

On this spinner, the red U, red Y, green Y, and blue U regions all have the same area, and the blue Y region has twice the area of the blue U region. In an experiment consisting of one spin, the set of possible outcomes has five members represented by the five regions.



{red U, red Y, green Y, blue U, blue Y}

Exercises 10-5a

(Class Discussion)

1. (a) On the spinner above, why are not all five outcomes equally likely?

   (b) Which outcomes are equally likely?

   (c) What probabilities do you suggest assigning to the five outcomes?

2. If you perform the experiment consisting of one spin, are you certain to get exactly one of these outcomes?

Now suppose we are interested in the likelihood of spinning red. However, "red" is not one of the five outcomes listed. We have red when we have either outcome "red U" or outcome "red Y". We shall speak of "red" in this case as an event. Thus we identify the event "red" with the subset of outcomes {red U, red Y}.

3. List the set of outcomes in the event "blue".

4. List the set of outcomes in the event "Y".

5. Consider the experiment of tossing a single die once.

   (a) List the set of possible outcomes.

   (b) Now consider the event "even number". List the subset of outcomes in this event.

   (c) List the outcomes in the event "odd number".

   (d) List the outcomes in the event "number greater than 4".

An event, therefore, is a subset of the set of possible outcomes. Recall that an outcome is sometimes called a simple event. All other events are built up out of simple events.

How can we use the probabilities of simple events to find the probabilities of other events?

For example, in the experiment that we have been considering, we noted that the event "red" is the subset of outcomes {red U, red Y}. We found that $P(\text{red } U) = P(\text{red } Y) = \frac{1}{6}$. Examining the spinner reveals that the fractional part that is colored red is $\frac{1}{3}$, hence $P(\text{red}) = \frac{1}{3} = \frac{1}{6} + \frac{1}{6}$. Thus we can write

$$P(\text{red}) = \frac{1}{3} = \frac{1}{6} + \frac{1}{6} = P(\text{red U}) + P(\text{red Y}).$$

As one more example, consider $P(Y)$. Of the area of the spinner, $\frac{2}{3}$ is lettered "Y", so we expect that $P(Y) = \frac{2}{3}$.

The event "Y" is the subset {red Y, blue Y, green Y}. The probabilities of the three outcomes in the set are

$$P(\text{red Y}) = \frac{1}{6}$$
$$P(\text{blue Y}) = \frac{1}{3}$$
$$P(\text{green Y}) = \frac{1}{6}.$$

Again, $P(Y) = \frac{2}{3} = \frac{1}{6} + \frac{1}{3} + \frac{1}{6} = P(\text{red Y}) + P(\text{blue Y}) + P(\text{green Y})$.

We thus see that:

> *An event is a subset of the set of possible outcomes. The probability of an event is the sum of the probabilities of the outcomes in it.*

In our examples, the sets of possible outcomes have had only a few members. We will soon see some more complicated situations in which there are many possible outcomes. For some experiments, the set of outcomes is infinite. In this chapter we will consider only situations in which we can use finite sets of outcomes.

We are now ready to summarize our development. In order to analyze an experiment and find probabilities, we proceed as follows:

(1) We choose a set of outcomes for the experiment. This set must be chosen so that the experiment is sure to result in *exactly one* of the outcomes.

(2) We assign to each outcome a probability. Each of these probabilities is a number between 0 and 1, and the sum of all of them is 1.

(3) Each event is a set of outcomes. The probability of an event is the sum of the probabilities of the outcomes in the event.

There is no single simple rule for deciding on a set of outcomes and deciding how to assign probabilities to them. Practice and experience will help improve your skill. We have noted the following remarks which serve as guidelines.

(a) If two or more outcomes seem to be equally likely, they should be assigned equal probabilities.

(b) If an event is certain, its probability should be 1.

Among the subsets of the set of possible outcomes, a special example is the empty set. Thus the empty set is an event. Since there are no outcomes in the event, we must understand what is meant by the "sum" of their probabilities. We interpret this "sum" as the number 0. Thus the empty set is an event whose probability is 0.

This remark fits with our earlier feeling that an event has probability 0 if the event can not occur. Since our experiment must result in an outcome, the empty set is an event that does not occur. We shall see further applications of this idea when we discuss intersections of events.

Exercises 10-5b

1. From an ordinary deck of 52 playing cards, one card is drawn.
   (a) How many possible outcomes are there?
   (b) List the subset of outcomes in the event "king".
   (c) What is the probability that the card drawn is a king?
   (d) What is the probability that the card drawn is a spade?
   (e) What is the probability that the card drawn is the queen of hearts?

2. One of the whole numbers from 1 through 30 (including 1 and 30) is selected at random; that is, the selection is made so that one number is just as likely to be chosen as any other.
   (a) List the subset of the outcomes in the event "prime number".
   (b) What is the probability that the number chosen is a prime number?
   (c) What is the probability that the number chosen is even?
   (d) What is the probability that the number selected is between 18 and 25?

3. Two black marbles and one white marble are in a box. Without looking inside the box, you are to take out one marble.

   (a) Find the probability that the marble will be black.

   (b) Find the probability that the marble will be white.

4. Suppose that you toss an honest coin 9 times.

   (a) Are you likely to get a head each time?

   (b) What is the probability that the coin will show a tail on the tenth toss?

   (c) Do the outcomes of the first 9 tosses have any effect on the outcome of the tenth toss?

5. Suppose that a box contains 48 marbles, of which 8 are black and 40 are white.

   (a) Find the probability that one marble picked without looking in the box will be white.

   (b) Find the probability that if nine marbles are taken out simultaneously all of them are black.

6. There are 25 students in a class, of whom 10 are girls and 15 are boys. The teacher has written the name of each pupil on a separate card. If the cards are shuffled and one card is drawn, what is the probability that the name written on the card is:

   (a) the name of a boy?

   (b) your name (assuming you are in the class)?

7. You have five playing cards: the ten, jack, queen, king, and ace of hearts. You draw them, one at a time, at random.

   (a) What is the probability that the first card you draw is the ace?

   (b) Assume that you draw the jack on the first draw, and put it aside. What is the probability that the second card you draw is the ace?

   (c) Why are your answers for (a) and (b) not the same?

   (d) After drawing the jack first, and putting it aside, assume that the second card you draw is the ten. Put that aside also. What is the probability that the third card you draw is the ace?

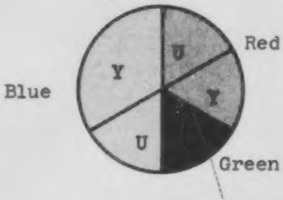
(e) What do you observe about the probabilities in parts (a), (b), and (d)?

8. In our discussions of the spinner pictured here, we selected as our set of possible outcomes {red U, red Y, green Y, blue U, blue Y}. Some other set might have been selected as the set of possible outcomes. For each of the following, decide whether it is an acceptable set of possible outcomes. If it is not, explain why not.

(a) {red, green, blue}

(b) {red, blue, Y}

(c) {U, Y}



9. For the spinner in Exercise 8 above, the following sets were proposed as sets of possible outcomes:

(i) {red U, red Y, green Y, blue U, blue Y}
(ii) {red, green, blue}
(iii) {red, blue, Y}
(iv) {U, Y}

(a) For each set decide whether or not the outcomes are equally likely.

(b) What probability would you assign to each outcome in each set above?

(c) For each of the sets, find the sum of the probabilities of the outcomes in that set.

(d) What do you notice about the sums for sets which are acceptable sets of possible outcomes?

(e) What do you notice about the sums for the other sets?

10-6. Counting Outcomes; Tree Diagrams

Exercises 10-6a

(Class Discussion)

1. Toss one coin twelve times and record the result of each toss. In this experiment let h be the number of times that the coin showed heads. Thus the possible values of h are 0, 1, 2, 3, ..., 10, 11, 12.

   (a) According to your previous experience, what value of h did you expect?

   (b) What value of h did you actually get?

2. Toss two coins together, a penny and a nickel. Record your result as an ordered pair by writing (side penny shows, side nickel shows). Repeat until you have recorded 12 pairs.

   (a) How many times did both coins show heads?

   (b) How many times did you expect both to show heads?

   (c) Was the number you stated in (b) about the same, or greater than, or less than the value of h that you stated in Exercise 1(a)?

   (d) How many times did you actually observe both coins showing tails?

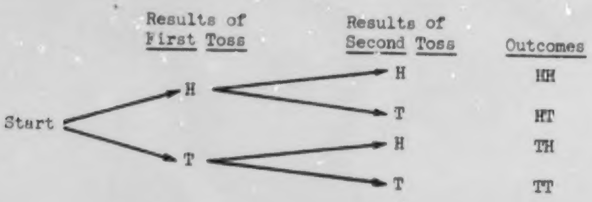
   (e) How many times did you observe the two coins showing one head and one tail?

In determining probability, we often have had occasion to list all of the possible outcomes of an experiment. In particular, we shall need a way of telling how many possible outcomes there are.

By way of illustration, suppose that we consider tossing a single coin. If we toss it once we have exactly two possible outcomes, heads and tails, which we shall denote by H and T respectively. If we toss the coin twice, we have four possible outcomes from the succession of tosses. We can show them in a table:

First Toss	Second Toss
H	H
H	T
T	H
T	T

The same information can be given in a "tree diagram," as pictured below:



The tree shows that, for each result of the first toss there are two results of the second toss. This is represented in the diagram by two arrows from each entry in the first column. The possible outcomes for the combination of two tosses can be seen by reading from left to right along the "branches of the tree". They are HH, HT, TH, TT. The number of outcomes is found by counting the ends of the branches at the right. Thus we can see that there are four outcomes without bothering to list them.

Exercises 10-6b

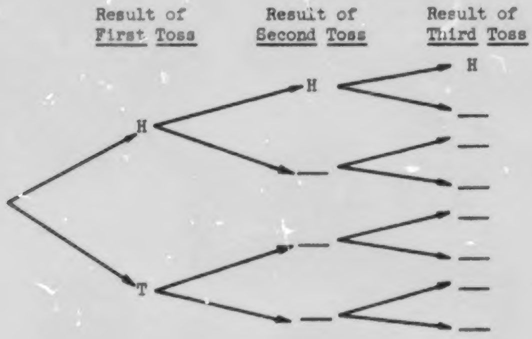
1. (a) Construct a tree diagram for the experiment of a single toss of a penny and a nickel.

   (b) What are the possible outcomes?

   (c) If the outcomes are equally likely, what probability would you assign to each?

   (d) How does the probability of HH compare with your results of 12 tosses of the two coins in Exercise 2 of Exercises 10-6a?

2. (a) Copy and complete this tree diagram for the experiment of tossing a coin three times.



(b) How many possible outcomes are there? How does this compare with the number of possible outcomes for two tosses? How many outcomes would you expect from a sequence of four tosses?

(c) Complete this list of the possible outcomes for three tosses:

HHH, HHT, ___, ___, ___, ___, ___, ___.

3. (a) Complete this table showing the number of tosses of a single coin, and the number of possible outcomes:

Tosses	Outcomes
1	2
2	4 = 2 × 2
3	8 = _____
4	__ = _____

(b) In general, if an experiment consists of tossing a coin n times, where n is a counting number, then the number of possible outcomes depends upon n, and can be written:

$$2 \times 2 \times 2 \times \ldots \times 2 \text{ with \_\_\_\_\_ factors.}$$

This number is more simply expressed with the exponent notation as:

$$2^{\_\_\_}.$$

(c) Use arrow notation to express this as a function T which assigns to any counting number n the number 2^n.

4. (a) Refer to the tree diagram you made for Exercise 2, and complete this table showing the number of heads involved in each outcome.

Outcome	Number of Heads
HHH	3
HHT	___
HTH	___
___	___
___	___
___	___
___	___
TTT	0

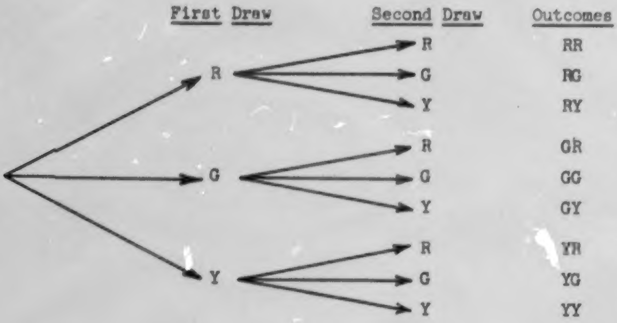
(b) Suppose that we let h represent the number of heads obtained. In how many cases does $h = 3$? does $h = 2$? does $h = 1$? does $h = 0$? In how many cases is it true that $h \geq 2$? that $h \geq 1$?

(c) Since we have 8 possible outcomes, all equally likely, what probability should we assign to each? What is the probability of getting:

three heads?
exactly two heads?
exactly one head?
no heads?
at least two heads?
at least one head?

In each exercise above we have studied situations where each trial had exactly two outcomes. The use of the tree diagram can often help in analyzing experiments for which the number of outcomes of a single trial is greater than two.

For example, suppose that we have a box containing 3 marbles alike except for color: 1 red, 1 green, 1 yellow. If we pick one marble at random, then there are three possible results, which we shall call R, G, and Y, for red, green, and yellow, respectively. If we return the chosen marble to the box and draw again, we again have 3 possibilities. Each outcome, for the experiment consisting of the pair of drawings, can be described in terms of the color on the first draw and the color on the second draw; for example, RY. The possibilities are shown in the tree diagram:



Exercises 10-6c

(Class Discussion)

1. In the tree diagram for the 3 marble experiment above,

   (a) How many possible results are there on the first draw?

   (b) For each result of the first draw, how many possibilities are there on the second draw?

   (c) How many possible outcomes are there for the succession of two drawings? How is this number related to your answers to (a) and (b)?

   (d) What probability would you assign to each outcome?

2. Suppose that after the second draw, the marble is again replaced and a third draw is made.

   (a) Construct a tree diagram showing all of the possible outcomes.

   (b) How many possible outcomes are there for the succession of three draws?

   (c) How is this number related to the number of marbles (3) and the number of drawings (3)?

(d) Make a table for the 3 marble experiment like the one in Exercise 3, Exercises 10-6b.

(e) Suppose an experiment consisted of n trials, where each trial consists of drawing one marble from the box containing 3 marbles, recording the color, and replacing the marble. How would you express the total number of possible outcomes, using exponents?

3. Consider an experiment which consists of tossing a single die several times.

(a) For one toss of the die, how many possible outcomes are there?

(b) For two tosses of the die, what is the number of possible outcomes for the succession of tosses?

(c) For three tosses of the die, what is the number of possible outcomes?

(d) Make a table for the die-throwing experiment like the one in Exercise 2(d) above.

(e) How can you express the total number of possible outcomes for n throws?

4. We have looked at several experiments, each consisting of n trials. Let us summarize what we have found.

(a) For coin tossing, (Exercises 10-6b), there are 2 possible outcomes for each toss, the number of possible outcomes keeps doubling so that for a succession of n tosses we have ______ possible outcomes.

(b) For the drawing of a marble from three marbles, (Exercises 1 and 2 above), there are 3 possible outcomes for each draw, hence for a succession of n draws we have ______ possible outcomes.

(c) For tossing a single die, each individual toss has ______ possible outcomes. For a succession of n tosses we have ______ possible outcomes.

The outcomes of throwing two dice can be thought of as a set of ordered pairs, made up of (number on first, number on second). We could make a tree diagram for this, but because of the rather large number of possible outcomes, there is a more compact way in which we can do it.

Exercises 10-6d

(Class Discussion)

1. In making a chart for the two-dice experiment, let us call the first die the red die and the second die the green die. Thus the ordered pair (5,2) indicates "5 on red and 2 on green".

(a) Complete the following table showing all possible outcomes of rolling the two dice:

		Green					
		1	2	3	4	5	6
Red	1	(1,1)	(1,2)	___	___	___	___
	2	(2,1)	___	___	___	___	___
	3	___	___	___	___	(3,5)	___
	4	___	___	(4,3)	___	___	___
	5	___	___	___	___	___	(5,6)
	6	___	(6,2)	___	___	___	___

Save this table for use later.

(b) How many possible outcomes are there? What probability would you assign to each outcome?

(c) How would you change the chart if, instead of rolling both dice together, you rolled the red die first, and then the green die?

(d) Make a table like the one in (a), but enter the sum of each ordered pair. That is, instead of (1,1), write "2".

(e) Make a table showing the frequency of each sum.

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	1										

2. Use the charts made in Exercise 1 to find the probability of each of the following events.

(a) The event "doubles".

(b) The event "sum is 9".

(c) The event "sum is less than or equal to 5".

3. (a) What sum is most likely to occur when two dice are tossed?

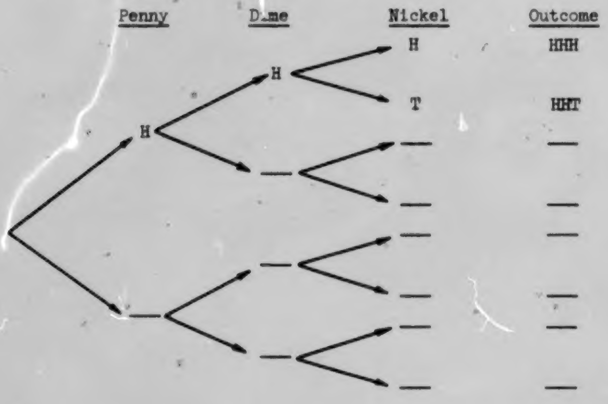
(b) Is its probability greater or less than $\frac{1}{4}$? $\frac{1}{5}$?

In this section and the two preceding ones we have found ways of assigning probabilities to the possible outcomes of an experiment, and of using the probabilities of outcomes to find the probability of an event involving two or more outcomes. In the experiments so far, we have dealt with outcomes that seemed equally likely, and with events which were subsets of the set of outcomes. We have used tables, charts, and tree diagrams as aids in counting outcomes.

Exercises 10-6e

1. Consider an experiment which consists of tossing three coins, a penny, a dime, and a nickel.

(a) Complete the tree diagram below, and list the possible outcomes.



(b) Assuming all coins are "honest", state the following probabilities for h, the number of heads:

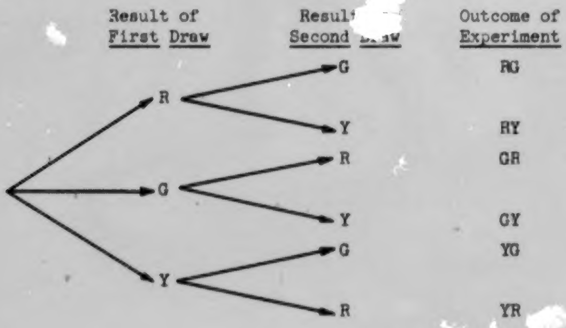
$P(h = 3)$ $\qquad$ $P(h = 0)$

$P(h = 2)$ $\qquad$ $P(h \geq 2)$

$P(h = 1)$ $\qquad$ $P(h \geq 1)$

(c) Compare your results for (a) and (b) with Exercises 2-4 in Exercises 10-6b, in which you used a tree diagram to analyze an experiment in which a single coin was tossed 3 times. What conclusion can you reach concerning the two experiments?

2. This time we consider another 3-marble experiment, still using a box containing one red, one green, and one yellow marble. We pick a marble at random, but this time we do not return it to the box before selecting another marble. The possible outcomes are shown by this tree diagram:



(a) What probability should we assign to each outcome?

(b) Compute the probabilities of events as indicated:

P(one marble red) = _____

P(second marble green) = _____

P(first marble yellow) = _____

(c) Make a similar chart showing the outcomes of drawing three times without replacing marbles, assign probabilities to the outcomes, and compute the following:

P(one marble red) = _____

P(first marble red) = _____

P(first marble red and second marble green) = _____

P(two red marbles) = _____

3. A bowl contains 10 marbles, of which 5 are white, 3 are black, and 2 are red. We shall assume that they are identical in size, hence that each marble is equally likely to be picked if you reach into the bowl and take one marble without looking.

(a) What is the probability that you will pick a white marble in one draw?

(b) Assuming that you pick a white marble the first time and do not replace it, what is the probability that you will pick a black marble the second time?

(c) Assuming that you pick a white marble the first time and a black marble the second time and do not replace them, what is the probability that you will pick a red marble the third time?

4. An experiment consists of one toss each of an ordinary die and a coin.

(a) Make a tree diagram showing the possible outcomes. Does it matter which you list first, the outcomes for the die or the outcomes for the coin?

(b) Assign probabilities to the possible outcomes of the combined tosses, and compute the following:

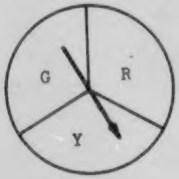
P(3,H) = _____

P(6,T) = _____

P(die shows a prime number) = _____

P(coin is tails) = _____

5. A game is played with a die and a spinner. The spinner shown at the right contains regions which have equal areas that are colored red, yellow and green. In the game,



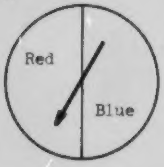
you spin once and throw the die once. What is the probability that you will spin red and then get 6 on the die?

10-7. Estimating Probabilities

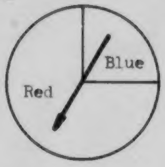
In many of the situations we have studied as examples (using coins, dice, spinners) it is possible to discuss the probability of certain events simply by thinking about the problem. We reason: If we have honest equipment and use it fairly, we can reach certain conclusions about it.

Example. Consider two spinners, I and II.

I



II



If the pointer is balanced and if it is honestly spun, we are willing to assert that:

$$\text{for Spinner I,} \quad P(\text{blue}) = \frac{1}{2} ;$$

$$\text{for Spinner II,} \quad P(\text{blue}) = \frac{1}{4} .$$

Notice that we assign these probabilities without actually spinning the pointer. We reason that, for Spinner I, the red and blue regions are equally likely if the spinner is "honest". Our reasoning is correct. How much this reasoning applies to a particular spinner can only be decided by actually experimenting many times. Suppose, after many trials, our results show approximately the same number of reds and blues. Then we are somewhat confident that the spinner is fair and that our reasoning is correct.

You may very well raise some questions regarding the last paragraph. For example:

(a) How many trials should be made?

(b) What precisely is meant by "approximately the same number" of reds or blues?

(c) How confident would we be?

These questions are all related and can only be answered somewhat generally here; our degree of confidence increases as we conduct more trials and as the fraction of red comes closer to $\frac{1}{2}$.

Exercises 10-7a

Suppose that a friend tells you that he has a spinner colored red and yellow. You cannot see the spinner. Your friend spins the pointer 30 times and tells you the result of each spin, which you record as follows:

RRYYR YRRYR RRRYR
YYRYR RRYRR RRYYR

1. How many of the 30 spins yielded red?
2. What probability can you estimate for the outcome of red on each spin?
3. What would you guess to be the relation of size of the red region to the size of the yellow region?
4. If you were told that you could win a prize by predicting correctly the color on the next spin, which color would you choose?
5. Suppose that many more trials are made and recorded-- 3000 trials in all. What would you guess about the relative areas of red and yellow if:
   (a) 1900 reds appeared?
   (b) 1512 reds appeared?

The problem that we have been discussing--where we know the results of performing the experiment a certain number of times but do not know the exact design of the spinner--is illustrative of many real life situations. There are many examples where decisions are made on the basis of estimated probabilities. These estimates, in turn, are based on past experience. Here are two examples.

(a) In a baseball game Robinson comes in as a relief pitcher. The opposing manager then orders Jones to bat for Smith. His decision is based on previous games, where experience has shown that Jones has had better success than Smith against Robinson. Regardless of the result in the present game, the manager may well claim that he is "playing percentage baseball".

(b) A doctor decides not to operate on Mrs. N. His decision is based on the fact that, in medical experience, a large percentage of the patients with her symptoms have been cured without undergoing expensive (and, perhaps, dangerous) surgery.

In a particular situation the confidence that is placed in a decision based on the results of previous trials depends both on the nature of the results and on the number of trials.

In the examples above, the idea of percent is applied to probabilities. Probabilities are often named as percents.

Recall that any rational number has many names. When a rational number expresses a ratio, it is frequently named as a percent. The symbol "%" means $\frac{1}{100}$. We can find the percent name by using the property of 1 to name the number by a fraction with denominator 100. Thus,

$$\frac{3}{4} = \frac{3}{4}\cdot\frac{25}{25} = \frac{75}{100} = 75\cdot\frac{1}{100} = 75\%.$$

$$\frac{1}{3} = \frac{1}{3}\cdot\frac{33\frac{1}{3}}{33\frac{1}{3}} = \frac{33\frac{1}{3}}{100} = 33\frac{1}{3}\cdot\frac{1}{100} = 33\frac{1}{3}\%.$$

Alternatively, we can use the division algorithm.

$$\frac{248}{531} = .467 = 46.7(.01) = \text{[illegible]}\%.$$

$$\begin{array}{r} .467 \\ 531\,\overline{)248.000} \\ \underline{212\ 4} \\ 35\ 60 \\ \underline{31\ 86} \\ 3\ 740 \\ \underline{3\ 717} \\ 23 \end{array}$$

Exercises 10-7b

1. From many thousands of manufactured articles of a certain type, the company selected a sample of 100 articles at random. These were very carefully tested and it was found that 98 of the articles met the desired standard.

   (a) What is the ratio of the number of satisfactory articles to the number of articles tested?

(b) What percent of the articles tested were satisfactory?

(c) What is the estimated probability that an article made by this process is up to standard?

(d) Why is it more practical to test a sample of the articles than to test all of those manufactured?

2. A random sample of 500 patients with a certain disease were treated with a new drug. Of these, 380 were helped.

(a) What is the ratio of the number helped to the number treated, expressed as a fraction in simplest form?

(b) Express the same ratio as a fraction whose denominator is 100. What percent of those treated were helped?

(c) What is the estimated probability that a given person with this disease will be helped by the new drug?

(d) If 4000 patients were treated with the drug, about how many would you expect to be helped?

3. In baseball a player's batting average is computed by dividing the number of his hits by the number of his official times at bat. It is usually expressed as a 3-place decimal.

(a) After 240 times at bat, a certain player has 72 hits. What is his batting average?

(b) What is a reasonable estimate of the probability that he will get a hit on his next time at bat?

(c) If he comes to bat 180 more times during the season, about how many more hits might we expect him to get?

4. Bill Edwards is especially proud of his ability as a foul shooter. In one series of games he made 15 goals out of 20 attempts.

(a) What was his ratio of number of goals to number of attempts?

(b) In what percent of the attempts did he shoot goals?

(c) What would you estimate the probability to be that he shoots a goal on his next attempt?

(d) In the next series of games, he shot 22 goals out of 25 attempts. In what percent of the 25 attempts did he shoot goals? How does this compare with his percentage in the first series of games?

In the exercises above, we have used different ways of expressing the probability of an event: as a fraction, as a decimal, and as a percent. Actually, each of these is a different way of naming the same number.

The fraction $\frac{15}{20}$ and the decimal .75 name the same number as $\frac{3}{4}$. The percent symbol, %, is used to name the ratio of a number to 100. Thus 75% means $\frac{75}{100}$, or .75.

Exercises 10-7c

1. Each of the following percentages gives the estimated probability that a certain event will occur. Write a fraction in simplest form which also gives each probability.

   (a) 80%

   (b) 16%

   (c) 100%

   (d) 41%

   (e) $37\frac{1}{2}$%

   (f) 6.5%

   (g) $1\frac{1}{4}$%

   (h) $33\frac{1}{3}$%

2. Which of the following decimals could be the probability that some event will occur? Write each number as a percent.

   (a) .37

   (b) 1.05

   (c) .125

   (d) .0025

   (e) 3.25

3. Copy and complete the following table of probabilities.

	Fractional Form	Decimal Form	Percent Form
(a)	$\frac{3}{4}$	____	____
(b)	$\frac{4}{9}$	____ (to nearest thousandth)	____
(c)	$\frac{28}{50}$	____	____
(d)	$\frac{2}{3}$	____ (to nearest hundredth)	____
(e)	$\frac{17}{200}$	____	____

Suppose that we try estimating probabilities in an experiment for which we have reason to think that the outcomes are not equally likely. If a rivet (or a tack or a flat-headed screw) is tossed onto a flat surface, it may fall "up" like this: or it may fall "down" like this: .

Exercises 10-7d

(Class Discussion)

1. Think about tossing the rivet 50 times. Just by looking at it, guess how many times it will fall "up" and how many times "down".
2. Now toss your rivet (or similar object) onto a flat surface 50 times, recording the result each time.
3. How many times did you get "up"? How does this compare with your guess in part (a)?
4. On the basis of your result, what is an estimate of the probability that your rivet will land "up" on a single toss? Express the probability both as a fraction and as a percent.
5. Why is it better to base the estimated probability on the results of 50 tosses than on your guess in part (a)?

Tossing a rivet provides an example of a situation in which we have no way of determining the probability of each outcome by inspection. Certainly you would expect a broad-headed tack to fall "up" more often than you would a long screw with a small head. For a given rivet, whatever guess you make is not likely to be close. When the writers performed the experiment, they guessed 20 "ups" out of 50 tosses, but obtained only 9, so their guess was not very close.

Exercises 10-7e

1. The record of a weather station shows that in the past 120 days its weather prediction of "rain" or "no rain" has been correct 89 times. What is the probability that its prediction for tomorrow will be correct?
2. Life insurance and life annuity rates are based on tables of mortality. A table of mortality includes statistical data on 100,000 people who

4

were alive at age 10. The following are ten lines from the Actuaries Table of Mortality.

Age	Number living	Number dying during next year	Age	Number living	Number dying during next year
10	100,000	676	40	78,653	815
12	98,650	672	50	69,517	1,108
13	97,978	671	60	55,973	1,698
14	97,307	671	70	35,837	2,327
21	92,588	683	99	1	1

According to the table, 676 of the 100,000 will not be alive at 11 years of age. Of the original 100,000, there are 97,978 alive at age 13, but 671 of these persons die within one year.

(a) How many of the original 100,000 were alive at the age of 50?

(b) What do you estimate the probability to be that a person who is 10 years old will live to the age of 50?

(c) Approximately what percent of the original 100,000 were alive at the age of 50?

3. (a) How many people of the original 100,000 were alive at age 40?

(b) Of those alive at age 40, what fraction were alive at age 50? About what percent?

(c) What is the estimated probability that a person who is 40 years old will live to the age of 50?

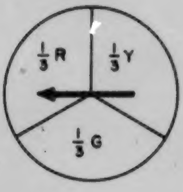
4. (a) What is the probability that a boy who is 10 years of age will live to the age of 99?

(b) What is the probability that a man who is 50 years of age will live to the age of 99?

(c) Who has the better chance of living to the age of 99, a boy of 10 or a man of 50?

10-8. Probability of A or B

Consider an experiment which consists of spinning the spinner shown here and tossing an ordinary die. Suppose that we need to find the probability that the color on the spinner is red or the number on the die is 4. -



Exercises 10-8a

(Class Discussion)

1. Complete this chart, showing the set of outcomes of the experiment.

		Number on Die					
		1	2	3	4	5	6
Color on Spinner	Red	(R, 1)	(R, 2)	(R, 3)	( )	( )	( )
	Yellow	(Y, 1)	(Y, 2)	( )	( )	( )	( )
	Green	(G, 1)	( )	( )	( )	( )	( )

2. If all outcomes are equally likely, what is the probability of each?

3. (a) Write set A, the set of outcomes in which the color is red.

(b) P(A) = ___?___

4. (a) Write set B, the set of outcomes in which the number is 4.

(b) P(4) = ___?___

5. (a) Write $A \cup B$. This means that we are looking for the set of outcomes in which either the color is red or the number is four.

(b) $P(A \cup B)$ = ___?___

(c) Why is the number of elements in $A \cup B$ not equal to the number of elements in A plus the number of elements in B?

6. (a) Write $A \cap B$. This means that we are looking for the set of outcomes in which the color is red and the number is four.

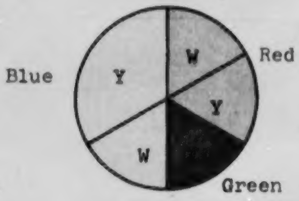
(b) $P(A \cap B)$ = ___?___

7. (a) Show that the number of elements in $A \cup B$ is equal to the number of elements in A plus the number of elements in B

minus the number of elements in $A \cap B$.

(b) Show that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

If we are given two sets A and B, will it always be true that the number of elements in $A \cup B$ equals the number in A plus the number in B minus the number in $A \cap B$? Consider a spinner divided as shown here. Suppose that you play a game where you win if the pointer lands on blue or on W. Verify the following chart which shows several events and the number of outcomes in each. (Note: By $n(A)$ we mean the number of elements in set A).

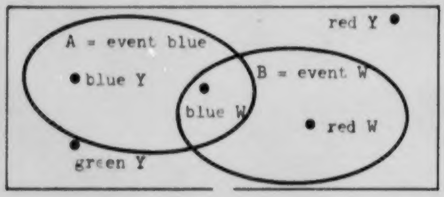


Event	Set of Outcomes in Event	Number of Outcomes
Blue	$A = \{\text{blue W, blue Y}\}$	$n(A) = 2$
W	$B = \{\text{blue W, red W}\}$	$n(B) = 2$
Blue or W	$A \cup B = \{\text{blue W, blue Y, red W}\}$	$n(A \cup B) = 3$
Blue and W	$A \cap B = \{\text{blue W}\}$	$n(A \cap B) = 1$

Thus again we see that

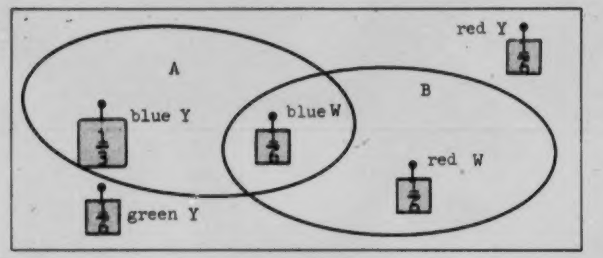
$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

We can show this on a Venn diagram as follows:



Since there is one element, the blue W, that is in both set A and set B, if we add n(A) and n(B) we shall have counted that element twice, hence we must subtract 1.

If we assign to each outcome its probability, then the Venn diagram can include that information also, and we have:



Check that: $P(A \cup B) = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$

$P(A) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$

$P(B) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P(A \cap B) = \frac{1}{6}$

$\frac{2}{3} = \frac{1}{2} + \frac{1}{3} - \frac{1}{6}$, or

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Exercises 10-8b

1. Let us think of throwing two dice, one red and one green. Either make a chart showing the possible outcomes or use the charts you completed for Exercises 10-6d to help you answer these questions.

   (a) If A is the event "the sum of the numbers is 6", state set A.

   (b) If B is the event "both numbers in the pair are even numbers", state set B.

   (c) State $A \cap B$.

   (d) Compute $P(A)$, $P(B)$, and $P(A \cap B)$.

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(e) Use the relation $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to find $P(A \cup B)$.

(f) Thus the probability, on a single toss of two dice, that the sum will be six or that both dice will show even numbers is _____.

2. The principal of Jones Junior High School said: "60% of the students in my school are boys; 15% of the students in the school play in the band. The number of boys who play in the band is 10% of the total number of students." If a student is chosen at random, what is the probability that the student is either a boy or a band member?

(a) If A is the event "boy is chosen", then $P(A) =$ _____.

(b) If B is the event "band member is chosen", then $P(B) =$ _____.

(c) Then $A \cap B$ is the event that a boy band member is chosen. $P(A \cap B) =$ _____.

(d) The probability that either a boy or a band member is chosen is P(_____).

(e) $P(A \cup B) =$ _____.

3. On a single toss of an ordinary die, find the probability that the number is either greater than 3 or an even number.

4. On a single toss of two dice, find the probability that the sum of the numbers thrown is either an odd number or a prime number.

5. On a single throw of two dice, find the probability that the sum of the numbers thrown is either a prime number or a "perfect" number. (A perfect number is a number such that the sum of all its factors, except the number itself, is the number.) Now, $6 = 3 \cdot 2 \cdot 1$, and $6 = 3 + 2 + 1$, so 6 is a perfect number, and is the only perfect number less than or equal to 12.

(a) If A is the event "a prime number", then $P(A) =$ _____.

(b) If B is the event "a perfect number", then $P(B) =$ _____.

(c) $P(A \cap B) =$ _____.

(d) $P(A \cup B) =$ _____.

In the exercise above, you found that $P(A \cap B) = 0$. This is because there is no possible way in which you could throw the perfect number 6 and also throw one of the prime numbers 2, 3, 5, 7, 11. Thus the intersection of A and B has no members. Since $A \cap B = \emptyset$, it follows that $P(A \cap B) = 0$.

Two events A and B are said to be mutually exclusive (or disjoint) if the occurrence of either excludes that of the other; they cannot both occur.

Exercises 10-8c

(Class Discussion)

1. If E and F are two mutually exclusive events, then $E \cap F =$ _____, and $P(E \cap F) =$ _____.

2. If a single card is drawn from a deck of cards, drawing an ace and drawing a jack _____________ (are, are not) mutually exclusive.

3. Drawing an ace and drawing a spade _____________ (are, are not) mutually exclusive events.

4. On one toss of 2 coins, the event "one head and one tail" and the event "two heads" _____________ (are, are not) mutually exclusive.

5. (a) On one toss of two dice, if E is the event "the sum is even" and F is the event "the sum is divisible by 3", E and F _____________ (are, are not) mutually exclusive.

   (b) $P(E) =$ _____, $P(F) =$ _____, $P(E \cap F) =$ _____, $P(E \cup F) =$ _____.

6. (a) Drawing a queen and drawing a king on one draw from a regular deck of cards _____________ (are, are not) mutually exclusive events.

   (b) P(king) = _____. (There are 52 equally likely outcomes and 4 of them are in the event "king".)

   (c) P(queen) = _____.

   (d) P(queen and king) = _____.

   (e) P(queen or king) = _____.

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7. The probability of throwing the sum 6 with two dice is $\frac{5}{36}$. If you throw two dice, what is the probability of not throwing a 6? Think about this before you go on.

   (a) $P(A) =$ ____; $P(A \cup C) =$ ____. (It is certain that the sum either is 6 or is not 6.)

   (b) A and C are mutually exclusive, so that $P(A \cap C) =$ ____.

   (c) Hence $P(A \cup C) = P(A) + P(C)$

$$1 = \frac{5}{36} + P(C)$$

$$P(C) = 1 - \frac{5}{36} = \_\_\_\_ .$$

Exercise 6 above is an illustration of a general result that is very useful. If $P(A)$ is the probability of event A, and $P(\text{not-}A)$ is the probability that A does not occur, then

$$P(\text{not-}A) = 1 - P(A).$$

Notice that the event A and the event not-A are always mutually exclusive. A and not-A are often called complementary events.

Exercises 10-8d

1. (a) If a batter's probability of getting a hit is .325, what is his probability of not getting a hit?

   (b) If the probability that a student passes a test is .85, what is the probability that he fails?

   (c) If the probability that a certain manufactured article is defective is .017, then what is the probability that it is not defective?

2. Which of the following pairs of events are mutually exclusive?

   (a) In tossing one coin: throwing heads; throwing tails.

   (b) In throwing a single die: throwing an odd number; throwing a 3.

   (c) In throwing a single die: throwing a 6; throwing a 3.

3. In a bag there are 4 red, 3 white, and 2 blue marbles. One marble is picked at random.

   (a) What is the probability of picking a red marble?

   (b) What is the probability of picking a white marble?

   (c) What is the probability of picking either a red or a white marble?

   (d) What is the probability that the marble picked is neither red nor white?

4. The gum machine has just been filled with 100 balls of gum of assorted colors; there are 25 red, 15 black, and 20 each of yellow, green, and white. The balls are mixed thoroughly, so that the chance of getting one ball is as good as the chance of getting any other. You buy one ball from the machine. What is the probability that you get:

   (a) a red?

   (b) a yellow?

   (c) either a black or a green?

5. There are 3 boys and 2 girls in a group. Two of them are chosen at random to buy refreshments for a party.

   (a) In how many ways can the choice be made?
   (b) How many of the pairs consist of two boys?
   (c) How many pairs consist of two girls?
   (d) How many pairs consist of one boy and one girl?
   (e) What is the probability that two boys are selected?
   (f) What is the probability that a boy and a girl are picked?
   (g) What is the probability that at least one boy is selected?

6. Below are two sets of information.

   (i) Joe's mathematics teacher has taught 1600 students in the past 10 years. During this time, he has given A as a final grade to 132 students, and B as a final grade to 508 students.

   (ii) During the past 3 years, Joe has received as final grades in mathematics four A's, one B and one C.

   Which of the above sets of data would you consider more appropriate in estimating what final grade Joe might get in mathematics at the end of this semester? Why?

7. In a game, Mary observed that the probability of a certain event M was $\frac{2}{3}$. Cathy analyzed another event C, and observed that $P(C) = \frac{1}{3}$. Max arrived on the scene just in time to hear the discussion about the probabilities. Being quick on the draw, he made the observation that since $P(M) + P(C) = 1$, then M and C must be complementary events. Lars disagreed with Max, stating that M and C do not have to be complements. Who was correct, Max or Lars? Support your decision either by proving or by producing an example.

10-9. Probability of A and B

Suppose that a box contains a red, a green, and a yellow marble, and that an experiment is performed as follows:

The box is shaken and one marble is drawn without looking as you draw.

The color of the marble is recorded as R, G, or Y, and the marble is returned to the box.

The box is shaken again and a marble is drawn.

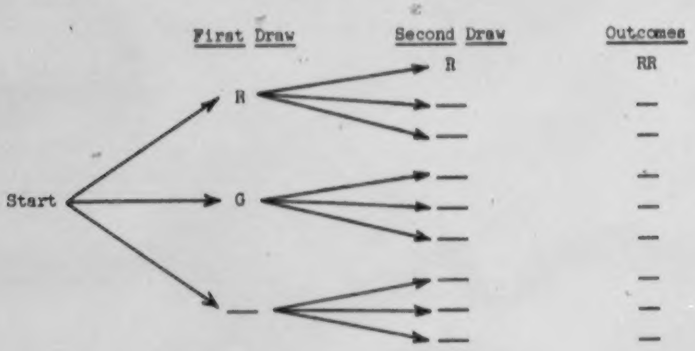
The color is recorded with the result of the first draw, to give a pair such as RR, RG, etc.

What do you think is the probability of getting red on both draws?

Exercises 10-9a

(Class Discussion)

1. Complete this tree diagram for the experiment.



2. How many outcomes are there in all? What probability should we assign to each outcome? What is P(RR)?

3. How many of the 9 outcomes are in the event "red on the first draw"? List them. What is the probability of red on the first draw? Which of the 9 outcomes are in the event "red on the second draw"? What is the probability of red on the second draw?

   What is the product of

   P(red on first) · P(red on second)?

4. How does P(red on first) · P(red on second) compare with P(red on first and second)?

5. If event A is "yellow on the first draw" and event B is "green on the second draw",

   (a) use the tree diagram in Exercise 1 to find $P(A)$, $P(B)$, and $P(A \cap B)$;

   (b) compare $P(A \cap B)$ with $P(A) \cdot P(B)$.

6. Consider another experiment, this one consisting of a throw of a die and a toss of a coin.

(a) Complete this chart, showing the possible outcomes.

		Number on Die					
		1	2	3	4	5	6
Result of Coin Toss	H	(H, 1)	(H, 2)	(___)	(___)	(___)	(___)
	T	(T, 1)	(T, 2)	(___)	(___)	(___)	(___)

(b) How many possible outcomes are there for this experiment? What probability would you assign to each of them?

(c) If E is the event that the number on the die is a composite number (either a 4 or a 6) and F is the event that the coin lands tails up, find $P(E)$, $P(F)$, and $P(E \cap F)$.

(d) Compare $P(E \cap F)$ with $P(E) \cdot P(F)$.

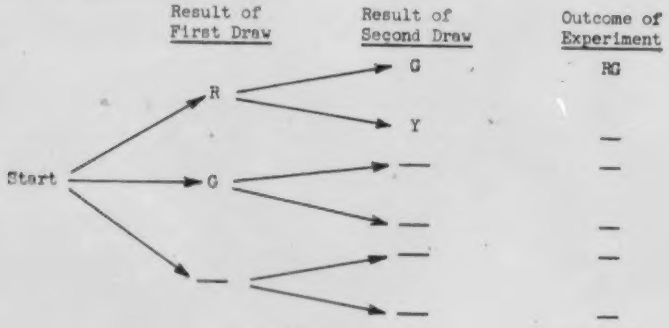
In each experiment, we have seen situations in which the probability of the intersection of two events is equal to the probability of one event multiplied by the probability of the other event. Do you think that this relationship holds in every case? Let us examine another situation.

Exercises 10-9b

(Class Discussion)

1. From the box containing three marbles, one each of red, green, and yellow, we again draw twice, but this time we do not replace the first marble drawn.

   (a) Compute this tree diagram.



(b) What is the probability of each of the six outcomes?

(c) If event A is "yellow on the first draw", which of the six outcomes are in A? What is $P(A)$?

(d) If event B is "green on the second draw", which of the six outcomes are in B? What is $P(B)$?

(e) From the list of outcomes, compute $P(A \cap B)$.

(f) Compare $P(A) \cdot P(B)$ with $P(A \cap B)$. Why are they not equal?

2. Now suppose the box contains three marbles, two blue and one yellow. The experiment will consist of drawing one marble, replacing it, then drawing a marble again. We wish to find the probability of the event, "two marbles of same color". In order to distinguish between the two blue marbles, we shall call them B_1 and B_2.

(a) Copy and complete this table showing the possible outcomes of the two-draw experiment.

Second Draw

		B_1	B_2	Y
First Draw	B_1	(B_1, B_1)		
	B_2			
	Y		(Y, B_2)	

(b) What is the number of possible outcomes?

(c) Draw a heavy horizontal line under the B_2 row in your table and a vertical line to the right of the B_2 column.

(d) What is P(two blue)? What is P(two yellow)? What is P(same color)?

(e) How do the lines you drew in (c) make it easy to answer the questions in (d)?

3. Now consider again the box containing 2 blue and 1 yellow marble. The experiment this time consists of drawing a marble twice, but without replacing the first marble drawn.

(a) Make a table of possible outcomes like the one in Exercise 2(a).

(b) How many possible outcomes are there? What outcomes in the 2(a) table are missing in this experiment?

(c) Draw lines as you did in Exercise 2(c).

(d) Find $P(\text{two blue})$, $P(2\text{ yellow})$, $P(\text{same color})$.

(e) If A is the event "blue on first draw", how many outcomes are in A? What is $P(A)$?

(f) If B is the event "blue on second draw", how many outcomes are in B? What is $P(B)$?

(g) What is the event "$A \cap B$"? What is $P(A \cap B)$? Compare $P(A \cap B)$ with $P(A) \cdot P(B)$.

Thus we find that the relationship

$$P(A) \cdot P(B) = P(A \cap B),$$

although true for some pairs of events A, B, does not hold for every pair. The truth of this equation depends upon the particular events A and B. This gives us a way of characterizing pairs of events, a way that will have considerable importance as you study more about probability and its applications.

Consider the two experiments in which we drew two marbles from a box. The first time we replaced the first marble before drawing the second. In this case, it seems natural to say that the two draws are *independent*. If we know the color of the first draw, this knowledge does not affect the probabilities that we assign to the possible second draw.

In the second experiment, in which the first marble was not put back, we recognize that knowing what happened on the first draw has a bearing on the probabilities assigned to the second draw. In this case the two drawc are *not* independent.

The decision concerning whether two events are independent is sometimes more complicated than these simple examples suggest. However, in certain experiments that involve two actions--like throwing a die and tossing a dime, or like drawing one marble and then another--we may usually rely on common sense to recognize independence. In such cases we feel intuitively that a pair of events are independent if the occurrence of either one does not affect the other. In case of doubt, we can consider that:

> In any experiment, events A and B are independent events if they satisfy the condition
>
> $$P(A \cap B) = P(A) \cdot P(B).$$

Exercises 10-9c

1. An experiment consists of throwing an ordinary die twice in succession. Consider the following four events:

   J, a composite number (4 or 6) on the first throw.

   K, a composite number (4 or 6) on the second throw.

   L, an even number on the first throw.

   M, an even number on the second throw.

   (a) Use the chart (Exercises 10-6d) showing all 36 possible outcomes and find:

   $P(J)$, $P(L)$,

   $P(K)$, $P(M)$.

   (b) Find $P(J \cap K)$.
   Compare $P(J) \cdot P(K)$ with $P(J \cap K)$.
   Are J and K independent events?

   (c) Find $P(J \cap L)$.
   Compare $P(J) \cdot P(L)$ with $P(J \cap L)$.
   Are J and L independent events?

   (d) Find $P(J \cap M)$.
   Compare $P(J) \cdot P(M)$ with $P(J \cap M)$.
   Are J and M independent events?

2. A jar contains five marbles which are alike except for color. Two are red, two are green, and one is blue. An experiment consists of drawing one marble at random from the jar and then drawing another marble at random, without having replaced the first.

   Consider these events:

   A, the first marble is green.

4-5

B, the second marble is green.

C, the second marble is blue.

(a) Which, if any, of these pairs of events do you believe are independent?

A and B

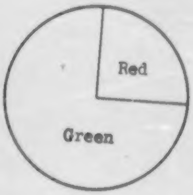
A and C

(b) We shall need to distinguish between two marbles of the same color, so we shall call them R_1, R_2, G_1, G_2, and B. Make a table like the one in Exercises 10-6d showing the twenty possible outcomes of two successive draws.

(c) Find $P(A)$, $P(B)$, $P(A \cap B)$.
Does $P(A \cap B) = P(A) \cdot P(B)$?
Are A and B independent events?

(d) Find $P(A)$, $P(C)$, $P(A \cap C)$.
Does $P(A \cap C) = P(A) \cdot P(B)$?
Are A and C independent events?

3. Suppose that you have a spinner like the one shown here, with one-fourth of the area red and three-fourths green.



(a) For one spin of the pointer there are two possible outcomes, R and G. Why do you feel that these outcomes are not equally likely? What numbers would you assign to $P(G)$? to $P(R)$?

(b) Make either a table or a tree diagram showing the possible outcomes of two spins.

(c) Which of the outcomes do you feel has the same probability as RG? Why is it reasonable to consider that "red on first spin" and "red on second spin" are independent events?

P(red on first spin) = ____

P(red on second spin) = ____

$P(RR) = P(R) \cdot P(R) =$ ____

(d) P(RG) = P(R) · P(G) = ____ · ____ = ____

P(GR) = P(RG) = ____

P(GG) = P(G) · P(G) = ____

(e) The sum of the probabilities of RR, RG, GR, GG should be _____. P(RR) + P(RG) + P(GR) + P(GG) =
____ + ____ + ____ + ____ = ____.

4. In a rural territory a county road crosses a state road at a dangerous intersection. Since the state road carries more traffic, the traffic light for the county road shows red three-fourths of the time and green one-fourth of the time. An agent on the county road drives across the state road each morning and returns each afternoon, at a random time. He wants to know the probability that on a given day he will get a green light both times, or one time, or not at all.

(a) As a model for this problem construct a spinner similar to the one in Exercise 3. What important change must you make?

(b) Draw a tree diagram or a table and assign probablities to the not equally likely outcomes, as was done before.

P(GG) = ____ P(RG) = ____

P(GR) = ____ P(RR) = ____

(c) How many times as likely is he to get a green light in the morning and red in the afternoon as he is to get green lights both times?

(d) Which is more likely, that he will get a red light both morning and afternoon or that he will get at least one green light?

(e) What percent of the time will he get one red light and one green light?

5. Suppose that you have a bag containing five black marbles and four white marbles.

(a) What is the probability of drawing two white marbles in succession from the bag, if the first marble drawn is replaced before the second drawing?

(b) What is the probability of drawing two white marbles if the first marble is not replaced before the second drawing?

6. There are 5 socks, unsorted, in a bureau drawer. Of these, 3 are blue and 2 are green. If you reach into the drawer in the dark and take out 2 socks, what is the probability that:

   (a) both are green?

   (b) both are blue?

   (c) one is green and one is blue?

7. In each of two bureau drawers you have some socks, not sorted into pairs. One drawer contains 9 socks, of which 5 are black and 4 are blue. The other drawer contains 7 black and 8 blue socks. If you pick one sock from each drawer without looking, what is the probability that:

   (a) both are black?

   (b) both are blue?

   (c) one is black and one is blue?

10-10. Summary

Section 10-2.

We frequently make decisions about what to do by guessing what is likely to happen when we cannot be certain. An objective of this chapter is to learn how to assign numerical measures to uncertain events. Such a measure is called the probability of the event.

Section 10-3.

A "game" played by two contestants is "fair" if the contestants have an equal chance to win. "Winning" means that certain events occur, "losing" that other events occur. Rules must make it certain that both cannot win, and that a player cannot both win and lose on the same play.

Looking at the records of the results of an experiment may help us to decide whether one event is more likely to occur than another.

Section 10-4.

In assigning measures to the probability of an event, the following guidelines are important.

(a) An event which is certain to occur has probability 1.
(b) An event which cannot occur has probability 0.
(c) Any other event has a probability between 0 and 1.
(d) Equally likely outcomes have equal probabilities.

Section 10-5.

In performing an experiment it is desirable to decide in advance on a set of possible outcomes. This must be done in such a way that, for any one trial in the experiment you are certain to get exactly one outcome out of the set of possible outcomes.

The sum of the probabilities of the set of possible outcomes of an experiment is 1.

An event is a subset of the set of possible outcomes. The probability of an event is the sum of the probabilities of the outcomes in the event.

Section 10-6.

The probability of an outcome can often be determined by counting the members of the set of possible outcomes. Tree diagrams and tables are helpful in determining the set of possible outcomes.

We find that if for a single trial in an experiment, there are two possible outcomes, then the number of possible outcomes for an experiment consisting of n such trials is 2^n.

Section 10-7.

In many situations in which probability is used to make predictions it is not possible to count the set of possible outcomes. In such cases, probabilities are often estimated on the basis of experimental evidence. A probability is sometimes expressed by a decimal or a percent, as well as by a fraction.

Section 10-8.

$P(A \cup B)$ is the probability that an event A or an event B (or both) will occur. This probability is related to $P(A)$, $P(B)$, and $P(A \cap B)$ by the equation

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Two events are mutually exclusive if, when one occurs, the other cannot occur. Event A and event not-A are always mutually exclusive. Such events are called complementary events, and $P(\text{not-}A) = 1 - P(A)$.

Section 10-9.

$P(A \cap B)$ is the probability that an event A and an event B both occur. If the occurrence of event A has no bearing on the occurrence of event B, then A and B are independent events. Independent events must satisfy the equation

$$P(A \cap B) = P(A) \cdot P(B).$$

Chapter 10

PROBABILITY

This chapter introduces the student to some fundamental ideas about probability. The theory of probability plays an increasingly important role in science, government, industry, business and economics. An understanding of the basic concepts of probability is essential for the study of statistical methods that are widely used in the behavioral and social sciences, as well as in the biological and physical sciences.

Probability is a mathematical subject about which most people have intuitive ideas. It turns out, however, that one's intuition can not always be trusted.

The mathematical background required for this chapter is not extensive. The examples used provide good opportunities for extending the student's understanding of rational numbers expressed in fraction, decimal and percent notation. The recording of data from experiments offers experience in orderly arrangement of information as a prerequisite to analysis of the data. Some familiarity with the language and notation of sets is needed; in particular, the concepts of union and intersection are used.

Experiments are an important part of the chapter. These are designed (1) to provide the student with data for examination and study; (2) to lead the student to develop a "feel" for probabilistic situations; and (3) to provide opportunity for the student to guess and perhaps formulate generalizations. The equipment needed includes coins, a deck of cards, dice, containers, colored marbles or disks; all these are traditional items for experiments in probability. Spinners are used also, because they are well suited for illustrating ideas in a simple and easily visualized way. They may be purchased or adapted from those provided with many children's games. Suppose, for example, that a spinner with ten congruent regions numbered 1-10 is available. It could be used, if necessary, when one which is $\frac{3}{4}$ red and $\frac{1}{4}$ green is called for. Simply call the 1 and 2 regions "green", the 3, 4, 5, 6, 7, and 8 regions "red", and ignore spins which result in 9 or 10.

The experiments are so designed that a student can use them working alone. In most cases, however, they can also be used by small groups of students working together and pooling their results.

Suggested Time Schedule

Section	10-1 10-2	10-3	10-4 10-5	10-6	10-7	10-8	10-9
Days	$\frac{1}{2}$	$2\frac{1}{2}$	3	2	3	2	2

Total: Three weeks

10-3. Fair and Unfair Games

The purpose of this section is to encourage the student to think objectively about chance events. Do not attempt at this time to assign numerical probabilities to the events in question. The rationale for such assignment comes in a later section.

Exercises 10-3b center around an experiment which may well be used as a project for a class period. Divide the class into four groups, give each a die and ask them to record the results of 25 tosses in the form suggested in Exercise 1. As each group reports its results, record them on the chalkboard. Then provide each student with forms for making the analyses of the results of the 100 tosses called for in Exercises 2, 3 and 4 (a tear sheet is provided in this commentary).

In addition to providing data for estimating the likelihood of certain events, these exercises are valuable in that they give the students experience in obtaining and organizing a mass of data which mean nothing until they have been arranged in an orderly fashion.

Exercises 10-3a (Class Discussion)

1. (a) Fair. There is exactly one face marked 1 and one marked 3. All other results yield a tie.

   (b) Fair. There are three odd-numbered faces (1, 3, 5) and three even-numbered faces (2, 4, 6).

(c) Not fair. You win only if 3 is thrown; he wins whenever 4, 5, or 6 is thrown. He has more "chances" to win than you do.

2. (a) Not fair. You have an advantage--there are three faces marked "3", but only one face marked "1".

   (b) Fair. Three faces are marked "3", and three faces are marked either "2" or "1".

   (c) Not fair. Only two faces (the ones marked "2") give you a win, while four faces favor him.

3. (a) Fair. Most throws will result in a tie, but throwing two 1's is as likely as throwing two 5's.

   (b) Fair. This game takes a bit of thought. Since it does not matter what happens to the green die, this game is essentially the same as the game described in 1(b) above.

   (c) The rule does not enable us to decide who should win if the dice fall with a green 4 and a white 6 at the same time. Strictly speaking, the "game" is not defined. Notice that if we agree to a tie when this situation occurs, then we have a true game and it is fair.

   (d) Not fair. You win only on the throw white 1, green 1. He wins on two throws: white 1, green 2 and white 2, green 1.

   (e) Not fair. You might list all the possibilities for the two dice (there are 36 of them). You will notice that you win on only 15 of them, while he wins on 21.

4. (a) Unfair. The situation is the same is in 3(e) above.

   (b) Fair. The chance of both even is the same as the chance of both odd.

Exercises 10-3b (Class Discussion)

1. Answers will vary.

2. Answers will vary.

3. Answers will vary.

4. (a) and (b) It could happen, but is very unlikely.

   (c) The die probably had no "4", "5", or "6" on any face. Actually, the row of numbers was made from the first row of numbers in Exercise 1 by assigning numbers as follows:

$$\begin{array}{ccccccccccccccccccccccccccccc}
4&3&5&5&3&&5&3&3&4&4&&1&4&1&6&6&&5&3&2&1&3&&4&6&4&5&1\\
\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\
2&3&3&3&3&&3&3&3&2&2&&1&2&1&3&3&&3&3&2&1&3&&2&3&2&3&1
\end{array}$$

   We were pretending that the die had one face with 1, two faces with 2 (the real 2, and the 4), and three faces with 3 (the real 3, the 5, and the 6). Incidentally, of these 25 throws we have: four 1's, seven 2's, and fourteen 3's. Recall that the die in Problem 2 of Exercises 10-3a has a 1 on one face, a 2 on each of two other faces, and 3's on the remaining faces. This is the kind of result you'd find with such a die. In fact, if you didn't have a die like this you could do an experiment about it anyway. You could simply use an ordinary die, and call the 4 "two", and the 5 and 6 "three", just as we did above.

4

Exercises 10-3c

1. (a)

	Number on die face					
	1	2	3	4	5	6
Frequency 1st row	4	1	6	6	5	3
Frequency 2nd row	3	1	7	3	7	4
Frequency 3rd row	2	5	8	4	4	2
Frequency 4th row	3	8	4	2	5	3
Total	12	15	25	15	21	12

(b) No, you would probably have expected the total frequency of each number to be the same. Since $\frac{1}{6} \times 100 = 16\frac{2}{3}$, you might expect each number to occur 16 or 17 times in 100 tosses. The 3 and the 5 occurred more frequently, and the 1 and the 6 much less frequently. The frequencies of the 2 and of the 4 were very close to the expected 16 or 17.

2. (a)

Frequency	Like Numbers	Consecutive Numbers
Exactly two	11	13
Exactly three	2	1
More than 3	0	0

(b) Answers may vary.

3. (a) Fair. The primes are 2, 3, 5 and the non-primes are 1, 4, 6.

(b) Unfair. The squares are 1, 4 and the non-squares are 2, 3, 5, 6.

10-4. Finding Probabilities

To carry out the experiments in this section you will need colored marbles and an opaque bag or other container (Exercises 10-4a), a die and a coin (Exercises 10-4b). We recommend that you use these experiments and base the discussion of basic concepts on the experience gained.

Three important ideas are developed:

(1) To describe the fact that two events are equally likely to occur, or that one is more likely than another, we use numbers. The symbol used to represent the probability of an event is $P(A)$, read "the probability of event A". Thus, $P(A)$ represents a number.

(2) The probability of an event which is certain to occur is defined to be 1, and the probability of an event which cannot occur is 0. All other events have probabilities between 0 and 1.

(3) Equally likely events have equal probabilities.

Exercises 10-4a (Class Discussion)

1. (a) The student would probably guess 4 .

   (b) Answers will vary. By combining the answers of all of the class, the total frequency of each color can be compared with $\frac{1}{3}$ of the total number of draws.

2. (a) Answers will vary. Again, the total frequency of each color when all results are combined will be of interest.

   (b) He will probably say that he would expect the total number of whites to be about twice the number of blues.

Exercises 10-4b

1. (a) 6 (b) 6 (c) yes (d) $\frac{1}{6}$

   (e) $P(1) = \frac{1}{6}$; $P(3) = \frac{1}{6}$; $P(5) = \frac{1}{6}$

   $P(2) = \frac{1}{6}$; $P(4) = \frac{1}{6}$; $P(6) = \frac{1}{6}$

2. (a) Two outcomes: heads, tails

(b) $\frac{1}{2}$

(c) $P(\text{heads}) = \frac{1}{2}$

$P(\text{tails}) = \frac{1}{2}$

(d) $\frac{1}{2}$

3. (a) $\frac{1}{6}$

(b) $\frac{1}{2}$ (You might, however, begin to suspect that the spinner is biased in some way.)

Exercises 10-4c

1. (a) 2, 4, and 6 (b) $\frac{1}{2}$

(c) $P(\text{even}) = \frac{1}{2}$ (d) $P(\text{odd}) = \frac{1}{2}$

(e) 1, 2

2. (a) 1 (b) $P(X) = 3 \cdot P(Y)$

(c) $P(Y) = \frac{1}{4}$, $P(X) = \frac{3}{4}$ (d) $P(X) + P(Y) = \frac{4}{4} = 1$

3. (a) $\frac{1}{7}$ (b) 1

(c) $P(K) + P(J) = 1 - \frac{4}{7} = \frac{3}{7}$

$P(K) = \frac{3}{7} - P(J) = \frac{3}{7} - \frac{1}{5} = \frac{8}{35}$

10-5. Outcomes and Events

Up to this point we have used the terms "trial", "experiment", "outcome", "event", relying on the context to convey their meaning. These terms have specific meanings in the theory of probability, and these meanings are developed in this section.

In particular, the distinction is made between "outcome" and "event". The distinction may be illustrated by using a die. If an experiment consists in a single toss of a die and recording the number of dots on the upper face, the set of possible *outcomes* is obvious.

$$A = \{1,2,3,4,5,6\}.$$

A toss will result in exactly one of these outcomes. These outcomes are equally likely, so the probability of any one of them is $\frac{1}{6}$.

We may be interested in whether the result of the toss is in a subset of these outcomes, rather than a particular number; for example, a number greater than 4. The members of A in this subset are

$$B = \{5,6\}.$$

We then speak of B as an *event*. That is, an *event* is a subset of the set of possible outcomes. Since the event B occurs whenever the outcome is 5 or the outcome is 6, the probability of event B is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Important understandings to be developed in this section, then, are these:

(1) Before an experiment is performed, the set of possible *outcomes* should be determined, and a theoretical probability assigned to each outcome. The set of outcomes must be chosen in such a way that in any trial in the experiment *exactly one* outcome occurs. The sum of the probabilities of the possible outcomes is 1. An outcome is sometimes called a *simple event*.

(2) Any subset of the set of possible outcomes is an *event*. The probability of an event is the sum of the probabilities of the outcomes in the event.

(3) One subset of the set of outcomes (as of *any* set) is the empty set. Since no one of the possible outcomes is a member of this empty set, its probability is 0.

Exercises 10-5a (Class Discussion)

1. (a) Because the blue Y region has twice the area of each of the other regions.

(b) Equally likely outcomes are: red U, red Y, green Y, and blue U.

(c) $P(\text{red } U) = P(\text{red } Y) = P(\text{green } Y) = P(\text{blue } U) = \frac{1}{6}$

$P(\text{blue } Y) = 2 \cdot P(\text{blue } U) = 2(\frac{1}{6}) = \frac{1}{3}$

2. Yes

3. {blue U,blue Y}

4. {blue Y,green Y,red Y}

5. (a) {1,2,3,4,5,6}

   (b) {2,4,6}

   (c) {1,3,5}

   (d) {5,6}

Exercises 10-5b

1. (a) 52

   (b) {king of clubs,king of diamonds,king of hearts,king of spades}

   (c) $P(\text{king}) = \frac{4}{52} = \frac{1}{13}$

   (d) $P(\text{spades}) = \frac{13}{52} = \frac{1}{4}$

   (e) $P(\text{queen of hearts}) = \frac{1}{52}$

2. (a) {2,3,5,7,11,13,17,19,23,29}

   (b) $P(\text{prime}) = \frac{10}{30} = \frac{1}{3}$

   (c) $P(\text{even}) = \frac{15}{30} = \frac{1}{2}$

   (d) $P(\text{number between } 18 \text{ and } 25) = \frac{6}{30} = \frac{1}{5}$

   (The set of numbers between 18 and 25 is {19,20,21,22,23,24}.)

3. (a) $P(\text{black}) = \frac{2}{3}$

   (b) $P(\text{white}) = \frac{1}{3}$

4. (a) No.

   (b) $\frac{1}{2}$

   (c) No.

5. (a) $P(\text{white}) = \frac{40}{48} = \frac{5}{6}$

(b) $P(9 \text{ black}) = 0$

6. (a) $P(\text{boy's name}) = \frac{15}{25} = \frac{3}{5}$

(b) $P(\text{your name}) = \frac{1}{25}$

7. (a) $\frac{1}{5}$

(b) $\frac{1}{4}$

(c) In (a) drawing the ace is one outcome out of five, while in (b) it is one out of four.

(d) $\frac{1}{3}$

(e) They are increasing.

8. (a) Acceptable.

(b) Not acceptable, since there is overlapping between the outcomes listed. Red and Y, or blue and Y, could happen simultaneously.

(c) Acceptable.

9. (a) (i) Not equally likely.

(ii) Not equally likely.

(iii) Not equally likely.

(iv) Not equally likely.

(b) (i) $P(\text{red U}) = \frac{1}{6}$, $P(\text{red Y}) = \frac{1}{6}$, $P(\text{green Y}) = \frac{1}{6}$, $P(\text{blue U}) = \frac{1}{6}$, $P(\text{blue Y}) = \frac{1}{3}$.

(ii) $P(\text{red}) = \frac{1}{3}$, $P(\text{green}) = \frac{1}{6}$, $P(\text{blue}) = \frac{1}{2}$.

(iii) $P(\text{red}) = \frac{1}{3}$, $P(\text{blue}) = \frac{1}{2}$, $P(Y) = \frac{2}{3}$.

(iv) $P(U) = \frac{1}{3}$, $P(Y) = \frac{2}{3}$.

(c) (i) $P(\text{red U}) + P(\text{red Y}) + P(\text{green Y}) + P(\text{blue U}) + P(\text{blue Y}) =$

$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = 1.$

(ii) $P(\text{red}) + P(\text{green}) + P(\text{blue}) = \frac{1}{3} + \frac{1}{6} + \frac{1}{2} = 1.$

(iii) $P(\text{red}) + P(\text{blue}) + P(Y) = \frac{1}{3} + \frac{1}{2} + \frac{2}{3} = 1\frac{1}{2}$.

(iv) $P(U) + P(Y) = \frac{1}{3} + \frac{2}{3} = 1$.

(d) The sums are all 1.

(e) The other sum is greater than 1.

10-6. Counting Outcomes; Tree Diagrams

The same kinds of materials are used for the experiments of this section--marbles, coins, and dice.

The students have already counted the possible outcomes for experiments in order to assign probabilities. In this section several ways are shown to facilitate the listing of outcomes in an orderly way. Constructing tree diagrams and tables are efficient ways to be sure all possibilities are listed.

Exercises 10-6a (Class Discussion)

1. (a) Answers may vary but will probably be 6 (or 5 or 7).

   (b) Answers depend on individual results.

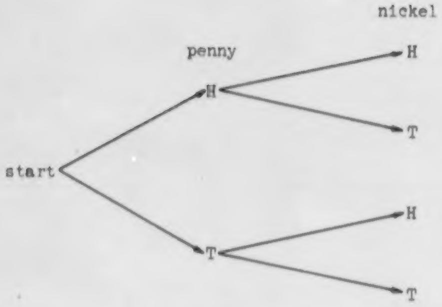
2. (a) Depends upon individual results.

   (b) Answers will vary--the teacher should discuss all that are offered. Actually, on one toss of two coins, the probability that both show heads is $\frac{1}{4}$, so the best answer would be 3 out of 12 pairs.

   (c), (d), and (e) Answers depend upon individual results.

Exercises 10-6b

1. (a)

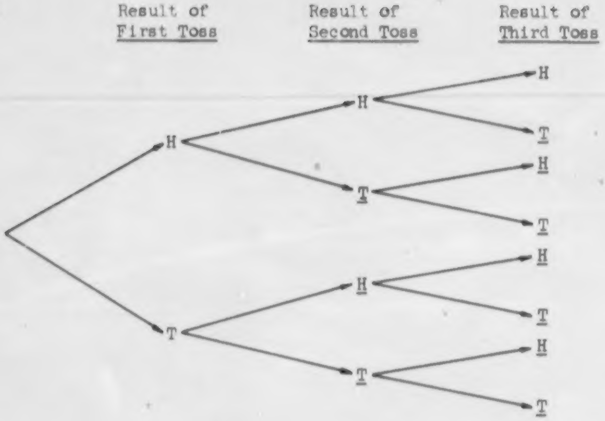

nickel
penny
H
H
T
start
H
T
T

(b) HH, HT, TH, TT

(c) $\frac{1}{4}$

(d) Answers will vary.

2. (a)


Result of First Toss
Result of Second Toss
Result of Third Toss
H
H
T
H
T
H
T
T
H
T
H
T
H
T

(b) There are 8 possible outcomes; this is twice as many as the number of possible outcomes for two tosses; for four tosses there should be 16 outcomes.

(c) HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

3. (a)

Tosses	Outcomes
1	2
2	$4 = 2 \times 2$
3	$8 = 2 \times 2 \times 2$
4	$16 = 2 \times 2 \times 2 \times 2$

(b) $2 \times 2 \times 2 \times \ldots \times 2$ with $\underline{n}$ factors;
expressed with exponent notation: 2^n

(c) $T : n \to 2^n$

4. (a)

Outcome	Number of Heads
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

(b) $h = 3$ once; $h = 2$ three times; $h = 1$ three times;
$h = 0$ once; $h \geq 2$ four times; $h \geq 1$ seven times.

(c) To each outcome, assign the probability $\frac{1}{8}$.

$P(3 \text{ heads}) = \frac{1}{8}$

$P(\text{no heads}) = \frac{1}{8}$

$P(\text{exactly 2 heads}) = \frac{3}{8}$

$P(\text{at least 2 heads}) = \frac{1}{2}$

$P(\text{exactly 1 head}) = \frac{3}{8}$

$P(\text{at least 1 head}) = \frac{7}{8}$

Exercises 10-6c (Class Discussion) are designed to develop the idea that if one trial in an experiment has 2 possible outcomes, then repetition of the trial n times will yield 2^n possible outcomes; if there are three possible outcomes for one trial, then there are 3^n outcomes for n trials; for a trial with 6 outcomes, then there are 6^n outcomes for n trials. This information is collected in Exercise 4. If students should wonder about the pattern which seems to be developing, you might encourage them to formulate a general result for a trial which has p possible outcomes. If it is repeated n times there will be p^n possible outcomes. This opportunity for noting the natural mathematical tendency to generalize results should be used only if it seems appropriate in the discussion. It is not important to saddle the students with a general formula to remember at this time.

Exercises 10-6c (Class Discussion)

1. (a) 3 possible results on first draw

   (b) For each of these, 3 possibilities on second draw

   (c) 9 possible outcomes for the succession of two draws; this is the product of the results of (a) and (b).

   (d) Each outcome has probability $\frac{1}{9}$.

2. (a) Here is the tree diagram for 3 draws from a box containing 3 marbles, 1 red, 1 yellow, 1 green, assuming each marble picked is returned to the box before the next draw is made.

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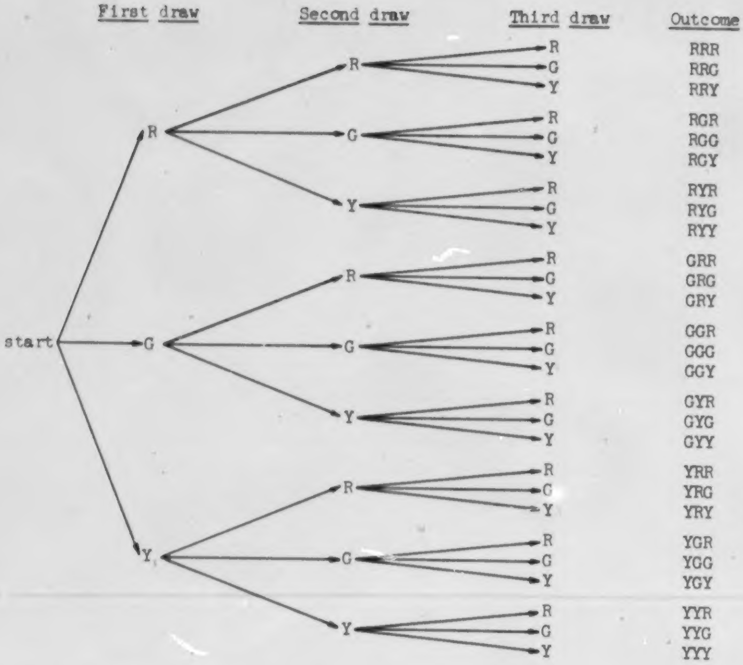

First draw
Second draw
Third draw
Outcome
start
R
G
Y
RRR
RRG
RRY
RGR
RGG
RGY
RYR
RYG
RYY
GRR
GRG
GRY
GGR
GGG
GGY
GYR
GYG
GYY
YRR
YRG
YRY
YGR
YGG
YGY
YYR
YYG
YYY

(b) 27 outcomes

(c) $3^3 = 27$. The base 3 is the number of marbles; the exponent 3 is the number of draws.

(d)

Tosses	Outcomes
1	3
2	$9 = 3 \times 3$
3	$27 = 3 \times 3 \times 3$
4	$81 = 3 \times 3 \times 3 \times 3$

(e) 3^n

3. (a) 6 possible outcomes

(b) $6^2 = 36$ possible outcomes

(c) $6^3 = 216$ possible outcomes

(d)

Tosses	Outcomes
1	6
2	$36 = 6 \times 6$
3	$216 = 6 \times 6 \times 6$
4	$1296 = 6 \times 6 \times 6 \times 6$

(e) 6^n

4. (a) 2^n

(b) 3^n

(c) 6; 6^n

Exercises 10-6d (Class Discussion)

1. (a)

		Green					
		1	2	3	4	5	6
Red	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

[Note: Students will use this table again in a later section.]

(b) 36; $\frac{1}{36}$

(c) No change

(d)

	Green					
Red	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

(e)

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	2	3	4	5	6	5	4	3	2	1

2. (a) P(doubles) $= \frac{6}{36} = \frac{1}{6}$

   (b) P(sum is 9) $= \frac{4}{36} = \frac{1}{9}$

   (c) P(sum ≤ 5) $= \frac{10}{36} = \frac{5}{18}$

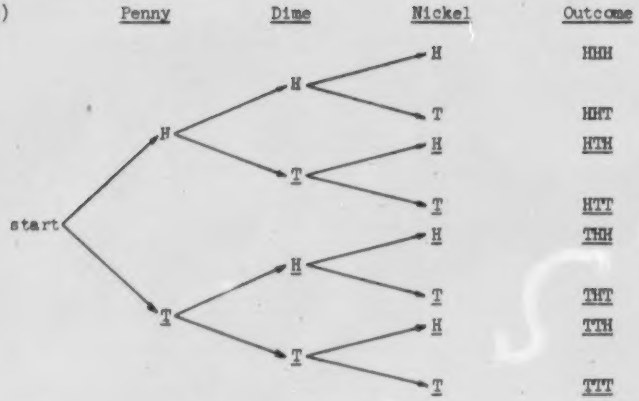
3. (a) Seven

   (b) P(seven) $= \frac{6}{36} = \frac{1}{6}$; $\frac{1}{6} < \frac{1}{4}$; $\frac{1}{6} < \frac{1}{5}$

In Exercises 10-6e some of the experiments begin development of the probability of unions and intersections, ideas which are considered more fully in Sections 10-8 and 10-9. Do not mention this to the students but be sure they understand the five problems.

Exercises 10-6e

1. (a)


Penny
Dime
Nickel
Outcome
start
H
T
H
T
H
T
H
T
H
T
H
T
H
T
HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

(b) $P(h = 3) = \frac{1}{8}$ $\quad$ $P(h = 0) = \frac{1}{8}$

$P(h = 2) = \frac{3}{8}$ $\quad$ $P(h \geq 2) = \frac{1}{2}$

$P(h = 1) = \frac{3}{8}$ $\quad$ $P(h \geq 1) = \frac{7}{8}$

(c) Tossing 1 coin three times has the same set of possible outcomes as tossing 3 coins once. The probabilities are the same in both experiments.

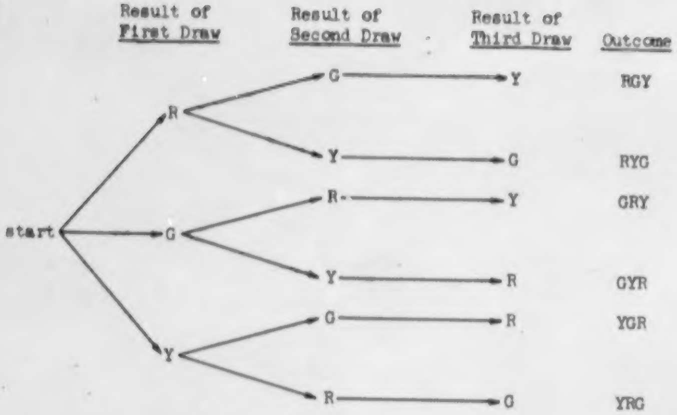
2. (a) $\frac{1}{6}$

(b) P(one marble red) = $\frac{4}{6} = \frac{2}{3}$

P(second marble green) = $\frac{2}{6} = \frac{1}{3}$

P(first marble yellow) = $\frac{2}{6} = \frac{1}{3}$

(c)


Result of First Draw
Result of Second Draw
Result of Third Draw
Outcome
start
R
G
Y
G
Y
R
Y
G
R
Y
G
Y
R
R
G
RGY
RYG
GRY
GYR
YGR
YRG

$P(\text{one marble red}) = \frac{6}{6} = 1$

$P(\text{first marble red}) = \frac{2}{6} = \frac{1}{3}$

$P(\text{first marble red and second marble green}) = \frac{1}{6}$

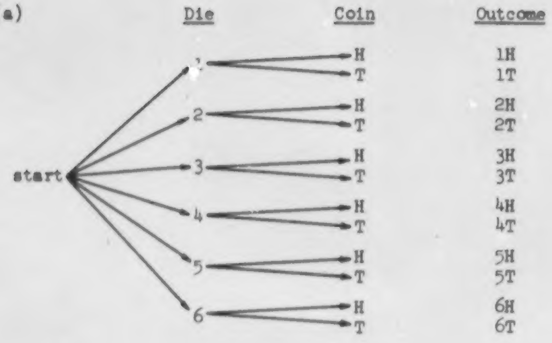
$P(\text{two red marbles}) = \frac{0}{6} = 0$

3. (a) $P(\text{white}) = \frac{5}{10} = \frac{1}{2}$

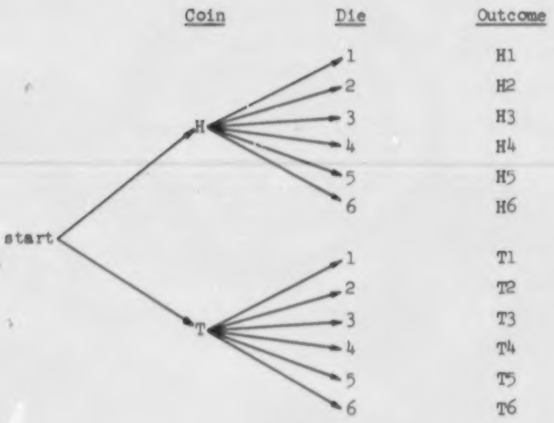
(b) $P(\text{black on second draw}) = \frac{3}{9} = \frac{1}{3}$

(c) $P(\text{red on third draw}) = \frac{2}{8} = \frac{1}{4}$

4. (a)


Die
Coin
Outcome
start
1
2
3
4
5
6
H
T
1H
1T
2H
2T
3H
3T
4H
4T
5H
5T
6H
6T

or


Coin
Die
Outcome
start
H
T
1
2
3
4
5
6
H1
H2
H3
H4
H5
H6
T1
T2
T3
T4
T5
T6

Since the order of the number and letter in any outcome does not matter, the outcomes are the same.

(b) The probability of each outcome is $\frac{1}{12}$

$P(3,H) = \frac{1}{12}$

$P(6,T) = \frac{1}{12}$

$P(\text{die shows a prime number}) = \frac{6}{12} = \frac{1}{2}$

$P(\text{coin is tails}) = \frac{6}{12} = \frac{1}{2}$

5. Since there are 6 outcomes for the die, and 3 for the spinner, there are, in all, 18 possible outcomes. Of those, only one is (R,6) hence

$$P(R,6) = \frac{1}{18} .$$

10-7. Estimating Probabilities

In practical situations it is frequently not possible to assign a theoretical probability of an event by thinking about the situation. Sometimes probabilities are estimated on the basis of past experiences, as is done in determining insurance premiums and in interpreting batting averages. Sometimes the probability of an event is estimated by sampling, as in election polls and in the testing of articles as described in Exercises 10-7b, Exercise 1. Estimated probabilities are often expressed as percents. This section provides opportunity for reviewing concepts of percentage, but this should not be permitted to overshadow the chief idea of the section, estimation of probability by empirical methods.

In this section our purpose is to give the student a little feeling for empirical probabilities. The exercises deal intuitively with the "true" probability of a given event; with an "unbiased" estimate of that true probability, computed as the proportion of occurrences of the given event in some number of trials; and with approximations to the resulting estimate.

For example, in Exercises 10-7a, we never know the true P(red) because we are never shown the spinner. In 30 trials we estimate P(red) as $\frac{19}{30}$, and then guess that the true spinner is approximately $\frac{2}{3}$ red and $\frac{1}{3}$ yellow. In other words, we guess that the red area is approximately twice as large as the yellow one, while our best unbiased estimate of that ratio, after the 30 trials is

$$\frac{\frac{19}{30}}{\frac{11}{30}}, \text{ or } \frac{19}{11}.$$

Later, in Exercise 5, the estimate $P(\text{red}) = \frac{1900}{3000} = \frac{19}{30}$ is based on 3000 trials, so we should be much less willing to approximate _this_ unbiased estimate by $\frac{2}{3}$. To do so would imply that we felt for some reason that the number 1900 of observed occurrences of red was "off" by 100. We would be adding this bias because $\frac{2}{3}$ is a "nicer" fraction than $\frac{19}{30}$.

Exercises 10-7a

1. 19 reds
2. Estimate that $P(\text{red}) = \frac{19}{30}$
3. If $P(\text{red})$ were truly $\frac{19}{30}$, then $\frac{19}{30}$ of the spinner would be red and $\frac{11}{30}$ would be yellow. The red region would be $\frac{19}{11}$ times as large as the yellow region. As a _guess_, however, we would say that the red region is about twice as large as the yellow region.
4. You would be smart to choose red.
5. (a) Probably you would still _guess_ the area of red to be about twice that of the yellow.

   (b) You would be apt to guess that the areas would be about equal.

Exercises 10-7b

1. (a) $\frac{98}{100} = \frac{49}{50}$

   (b) 98%

   (c) $\frac{49}{50}$

   (d) It is less expensive in any case, and is the _only_ way feasible in cases where the test would destroy the articles being tested.

2. (a) $\frac{380}{500} = \frac{19}{25}$

(b) $\frac{76}{100}$; 76%

(c) $\frac{19}{25}$ (or .76, or 76%); an approximation to this might be $\frac{3}{4}$

(d) 76% of 4000 = 3040; $\frac{3}{4} \times 4000 = 3000$; thus either 3040 or 3000 would be a reasonable answer, depending on the interpretation of the word "about".

3. (a) .300

(b) .3 (or $\frac{3}{10}$)

(c) About 54 hits

4. (a) $\frac{15}{20}$ or $\frac{3}{4}$

(b) 75%

(c) $\frac{3}{4}$ (or 75%)

(d) 88%; better, since 88% > 75%

Exercises 10-7c

1. (a) $\frac{4}{5}$

(b) $\frac{4}{25}$

(c) $\frac{1}{1}$ or 1

(d) $\frac{41}{100}$

(e) $\frac{3}{8}$

(f) $\frac{13}{200}$

(g) $\frac{1}{80}$

(h) $\frac{1}{3}$

2. [(a), (c) and (d) could be the probability of some event.]

(a) 37%

(b) 105%

(c) 12.5% (or $12\frac{1}{2}$ %)

(d) .25% (or $\frac{1}{4}$ %)

(e) 325%

3.

	Fractional Form	Decimal Form	Percent Form
(a)	$\frac{3}{4}$	.75	75%
(b)	$\frac{4}{9}$	.444	44.4% (or $44\frac{4}{9}$ %)
(c)	$\frac{28}{50}$	.56	56%
(d)	$\frac{2}{3}$	.67	67% (or $66\frac{2}{3}$ %)
(e)	$\frac{17}{200}$	.085	8.5% (or $8\frac{1}{2}$ %)

Exercises 10-7d (Class Discussion)

1-4. Answers will vary.

5. In the long run, we would expect that the estimated probability based on a large number of experiments gets quite close to the true probability. Unless we have very strong reasons for our guesses in Exercise 1, 50 trials provide a much better estimate than 0 trials.

Exercises 10-7e

1. $\frac{89}{120}$ (or approximately .742)

2. (a) 69,517

(b) $\frac{69,517}{100,000}$ (or .695 approximately)

(c) 69.5%

3. (a) 78,653

(b) $\frac{69,517}{78,653}$; 88%

(c) 88%

4. (a) $\frac{1}{100,000}$ (or .00001)

(b) $\frac{1}{69,517}$ (or .0000144, approximately)

(c) a man of 50

10-8. Probability of A or B

In this section we deal with this question: If we know the probabilities of two events, how can we determine the probability that one or the other will occur? Since probabilities are usually assigned by counting, this question really goes back to a more basic one: Given two finite sets and the numbers associated with them, how do we find the number associated with the union of the two sets? This question is dealt with in Exercises 10-8a and the table figures in the discussion which follows. You may wish to review set union and intersection by using examples such as the following:

If $A = \{\text{Tom, Jim, Pete, Harry, Joe}\}$
and $B = \{\text{Jim, Mary, Pete, Anne}\}$
then $A \cup B = \{\text{Tom, Jim, Pete, Harry, Joe, Mary, Anne}\}$
and $A \cap B = \{\text{Jim, Pete}\}$.

Then $n(A) = 5$, $n(B) = 4$, and $n(A) + n(B) = 5 + 4 = 9$, $n(A \cup B) = 7$, which is

$$n(A) + n(B) - n(A \cap B),$$

the subtraction being necessary to compensate for the fact that Pete and Jim were counted twice, once in determining $n(A)$ and again in determining $n(B)$.

Exercises 10-8a (Class Discussion)

1\.

Number on Die

Color on Spinner		1	2	3	4	5	6
	Red	(R,1)	(R,2)	(R,3)	(R,4)	(R,5)	(R,6)
	Yellow	(Y,1)	(Y,2)	(Y,3)	(Y,4)	(Y,5)	(Y,6)
	Green	(G,1)	(G,2)	(G,3)	(G,4)	(G,5)	(G,6

2. $\frac{1}{18}$

3. (a) $A = \{(R,1),(R,2),(R,3),(R,4),(R,5),(R,6)\}$

(b) $P(A) = \frac{6}{18} = \frac{1}{3}$

4. (a) $B = \{(R,4),(Y,4),(G,4)\}$

(b) $P(4) = \frac{3}{18} = \frac{1}{6}$

5. (a) $A \cup B = \{(R,1),(R,2),(R,3),(R,4),(R,5),(R,6),(Y,4),(G,4)\}$

(b) $P(A \cup B) = \frac{8}{18} = \frac{4}{9}$

(c) Because $(R,4)$ is in A and also in B, and an element of a set is counted only once.

6. (a) $A \cap B = \{(R,4)\}$

(b) $P(A \cap B) = \frac{1}{18}$

7. (a) $n(A) = 6$; $n(B) = 3$; $n(A \cup B) = 8$; $n(A \cap B) = 1$

$n(A) + n(B) - n(A \cap B) = 6 + 3 - 1 = 8 = n(A \cup B)$

(b) $P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{6} - \frac{1}{18} = \frac{8}{18} = \frac{4}{9} = P(A \cup B)$

In Exercises 10-8b, Exercise 5 develops the probability of event A or event B when, if A happens, B cannot happen; that is, $A \cap B$ is the empty set. Such events are called mutually exclusive events.

Exercises 10-8b

1. (a) $A = \{(5,1),(4,2),(3,3),(2,4),(1,5)\}$

   (b) $B = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

   (c) $A \cap B = \{(4,2),(2,4)\}$

   (d) $P(A) = \frac{5}{36}$; $P(B) = \frac{1}{4}$; $P(A \cap B) = \frac{1}{18}$

   (e) $P(A \cup B) = \frac{5}{36} + \frac{9}{36} - \frac{2}{36} = \frac{12}{36} = \frac{1}{3}$

   (f) $\frac{1}{3}$

2. (a) $P(A) = .60$

   (b) $P(B) = .15$

   (c) $P(A \cap B) = .10$

   (d) $(A \cup B)$

   (e) $P(A \cup B) = .60 + .15 - .10 = .65$

3. $P(A) = P(n > 3) = \frac{1}{2}$; $P(B) = P(\text{even}) = \frac{1}{2}$

   $P(A \cap B) = \frac{2}{6} = \frac{1}{3}$; $P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$

4. $P(A) = P(\text{odd}) = \frac{1}{2}$; $P(B) = P(\text{prime}) = \frac{15}{36} = \frac{5}{12}$; $P(A \cap B) = \frac{14}{36} = \frac{7}{18}$

   $P(A \cup B) = \frac{1}{2} + \frac{5}{12} - \frac{7}{18} = \frac{19}{36}$

5. (a) $P(A) = \frac{5}{12}$

   (b) $P(B) = \frac{5}{36}$

   (c) $P(A \cap B) = 0$

   (d) $P(A \cup B) = \frac{15}{36} + \frac{5}{36} - \frac{0}{36} = \frac{20}{36} = \frac{5}{9}$

In Exercises 10-8c, Exercise 7 deals with the probability of complementary events. Note that complementary events are always mutually exclusive, but the converse is not necessarily true; that is, mutually exclusive events are not always complementary. Exercise 6 is a case in point: Drawing a king and drawing a queen (on a single draw) are mutually exclusive events, but not complementary events.

Exercises 10-8c

1. $E \cap F = \emptyset$; $P(E \cap F) = 0$
2. Are
3. Are not
4. Are [Note: On Exercises 4-6 we are considering a single outcome--one toss, drawing, etc.]
5. (a) Are not

   (b) $P(E) = \frac{1}{2}$; $P(F) = \frac{1}{3}$; $P(E \cap F) = \frac{1}{6}$

   $P(E \cup F) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$
6. (a) Are

   (b) $P(\text{king}) = \frac{1}{13}$

   (c) $P(\text{queen}) = \frac{1}{13}$

   (d) $P(\text{queen and king}) = 0$

   (e) $P(\text{queen or king}) = \frac{2}{13}$
7. (a) $P(A) = \frac{5}{36}$; P(A [illegible] C) = 1

   (b) $P(A \cap C) = 0$

   (c) $P(C) = \frac{31}{36}$

Exercises 10-8d

1. (a) $1 - .325 = .675$

   (b) $1 - .85 = .15$

   (c) $1 - .017 = .983$
2. (a) mutually exclusive

   (b) not mutually exclusive

   (c) mutually exclusive

3. (a) $P(\text{red}) = \frac{4}{9}$

(b) $P(\text{white}) = \frac{3}{9} = \frac{1}{3}$

(c) $P(\text{red } \underline{\text{or}} \text{ white}) = \frac{7}{9}$

(d) $P(\text{neither red } \underline{\text{nor}} \text{ white}) = P(\text{blue}) = \frac{2}{9}$

4. (a) $P(\text{red}) = \frac{25}{100} = \frac{1}{4}$

(b) $P(\text{yellow}) = \frac{20}{100} = \frac{1}{5}$

(c) $P(\text{black or green}) = \frac{15}{100} + \frac{20}{100} = \frac{35}{100} = \frac{7}{20}$

5. (a) 10 ways

(b) 3

(c) 1

(d) 6

(e) $\frac{3}{10}$

(f) $\frac{6}{10}$

(g) $\frac{3}{10} + \frac{6}{10} = \frac{9}{10}$

6. Set (ii), since Joe is directly involved in this; his past performance is a better predictor of his future performance than is the teacher's tendency to give A's and B's.

7. Lars was correct. The problem says that Cathy "analyzed another event, C". However, it need not have been the complement of event M. For example, suppose that the experiment consists of a toss of a die, and that M is the event "number > 2". Then $P(M) = \frac{2}{3}$. If C is the event "prime number > 2", then $P(C) = \frac{1}{3}$ and $P(M) + P(C) = 1$, but the complement of M is "number ≤ 2".

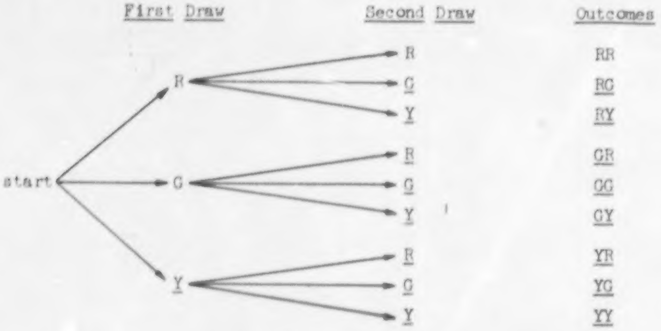
10-9. Probability of A and B

In Section 10-8 we considered the probability of $A \cup B$, that is, that event A or event B will occur. This section deals with $P(A \cap B)$, that is, that both A and B will occur. The cases developed first (Exercises 10-9a) lead to the conjecture that $P(A \cap B) = P(A) \cdot P(B)$. In Exercises 10-9c cases are shown in which the events A and B are related in such a way that the formula fails to hold. This leads to the definition for independent events, a concept which is very important for further study of probability. We can in many cases decide from the nature of the situation whether or not the fact that one event occurs has any bearing on the occurrence of a second event, but this is not always the case. Events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

Exercises 10-9a (Class Discussion)

1.



2. There are 9 outcomes; P(each outcome) $= \frac{1}{9}$

   $P(RR) = \frac{1}{9}$

3. The event "red on the first draw" contains 3 outcomes. They are RR, RG, RY. P(red on the first draw) $= \frac{3}{9} = \frac{1}{3}$.

   The event "red on the second draw" consists of the outcomes: RR, GR, YR. P(red on the second draw) $= \frac{3}{9} = \frac{1}{3}$.

   P(red on first) $\cdot$ P(red on second) $= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

4. P(red on first and second) $= \frac{1}{9}$, hence

P(red on first) $\cdot$ P(red on second) = P(red on first and second).

5. (a) $P(A) = \frac{1}{3}$; $P(B) = \frac{1}{3}$; $P(A \cap B) = \frac{1}{9}$

(b) $P(A \cap B) = P(A) \cdot P(B)$

6. (a)

		Number on Die					
		1	2	3	4	5	6
Result of Coin Toss	H	(H,1)	(H,2)	(H,3)	(H,4)	(H,5)	(H,6)
	T	(T,1)	(T,2)	(T,3)	(T,4)	(T,5)	(T,6)

(b) 12 outcomes. P(each outcome) $= \frac{1}{12}$

(c) $P(E) = \frac{4}{12} = \frac{1}{3}$; $P(F) = \frac{6}{12} = \frac{1}{2}$

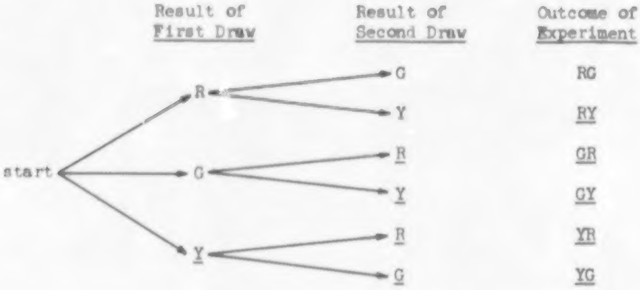
$P(E \cap F) = \frac{2}{12} = \frac{1}{6}$

(d) $P(E) \cdot P(F) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

Hence, $P(E \cap F) = P(E) \cdot P(F)$.

Exercises 10-9b (Class Discussion)

1. (a)


Result of First Draw
Result of Second Draw
Outcome of Experiment
start
R
G
Y
G
Y
R
Y
R
G
RG
RY
GR
GY
YR
YG

(b) P(each outcome) $= \frac{1}{6}$

(c) $A = \{YG, YR\}$; $P(A) = \frac{2}{6} = \frac{1}{3}$

(d) $B = \{RG, YG\}$; $P(B) = \frac{1}{3}$

(e) $\{A \cap B\} = \{YG\}$; $P(A \cap B) = \frac{1}{6}$

(f) $P(A) \cdot P(B) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

Hence, $P(A) \cdot P(B) \neq P(A \cap B)$.

They are not equal because the result of the first draw does affect what happens on the second draw, since the first marble drawn is not replaced before the second is drawn.

2. (a) and (c)

		Second Draw		
		B_1	B_2	Y
First Draw	B_1	(B_1, B_1)	(B_1, B_2)	(B_1, Y)
	B_2	(B_2, B_1)	(B_2, B_2)	(B_2, Y)
	Y	(Y, B_1)	(Y, B_2)	(Y, Y)

(b) 9

(c) See table.

(d) P(two blue) $= \frac{4}{9}$; P(two yellow) $= \frac{1}{9}$; P(same color) $= \frac{5}{9}$.

(e) The lines separate the table into four sections. The upper left section contains the outcomes in the event "two blue". The lower right section contains the outcomes in the event "two yellow".

3. (a) and (c)

		Second Draw		
		B_1	B_2	Y
First Draw	B_1		(B_1, B_2)	(B_1, Y)
	B_2	(B_2, B_1)		(B_2, Y)
	Y	(Y, B_1)	(Y, B_2)	

(b) 6. The outcomes (B_1, B_1), (B_2, B_2) and (Y, Y) are not possible because the marble drawn first was not returned to the box before the second draw.

(c) See table.

(d) P(two blue) $= \frac{2}{6} = \frac{1}{3}$; P(two yellow) = 0: P(same color) $= \frac{1}{3}$.

(e) 4. P(A) $= \frac{4}{6} = \frac{2}{3}$.

(f) 4. P(B) $= \frac{4}{6} = \frac{2}{3}$.

(g) "$A \cap B$" is the event of drawing two blue marbles.

$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$

$P(A) \cdot P(B) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$

$P(A \cap B) \neq P(A) \cdot P(B)$

Exercises 10-9c

Exercise 7 is harder than the rest and might well be omitted for an average class.

1. (a)

Number on Second Throw

Number on First Throw	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

[Note: Compare this with the chart for one throw of 2 dice in Exercises 10-6d.]

$P(J) = \frac{12}{36} = \frac{1}{3}$; $P(L) = \frac{18}{36} = \frac{1}{2}$

$P(K) = \frac{12}{36} = \frac{1}{3}$; $P(M) = \frac{18}{36} = \frac{1}{2}$

(b) $P(J \cap K) = \frac{4}{36} = \frac{1}{9}$; $P(J) \cdot P(K) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

$P(J) \cdot P(K) = P(J \cap K)$, hence J and K are independent events.

(c) $P(J \cap L) = \frac{12}{36} = \frac{1}{3}$; $P(J) \cdot P(L) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(J) \cdot P(L) \neq P(J \cap L)$, hence J and L are not independent events.

(d) $P(J \cap M) = \frac{6}{36} = \frac{1}{6}$; $P(J) \cdot P(M) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(J) \cdot P(M) = P(J \cap M)$, hence J and M are independent events.

2. (a) Answers will vary; we hope the student will reason that in neither pair are the events independent, since the experiment is of the "draw without replacement" type.

(b) Second Draw

First Draw	R_1	R_2	G_1	G_2	B
R_1		R_1, R_2	R_1, G_1	R_1, G_2	R_1, B
R_2	R_2, R_1		R_2, G_1	R_2, G_2	R_2, B
G_1	G_1, R_1	G_1, R_2		G_1, G_2	G_1, B
G_2	G_2, R_1	G_2, R_2	G_2, G_1		G_2, B
B	B, R_1	B, R_2	B, G_1	B, G_2	

(c) $P(A) = \frac{8}{20} = \frac{2}{5}$; $P(B) = \frac{8}{20} = \frac{2}{5}$; $P(A \cap B) = \frac{2}{20} = \frac{1}{10}$

$P(A) \cdot P(B) = \frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$

$P(A \cap B) \neq P(A) \cdot P(B)$, hence A and B are not independent events.

(d) $P(A) = \frac{2}{5}$; $P(C) = \frac{4}{20} = \frac{1}{5}$; $P(A \cap C) = \frac{2}{20} = \frac{1}{10}$

$P(A) \cdot P(C) = \frac{2}{5} \cdot \frac{1}{5} = \frac{2}{25}$

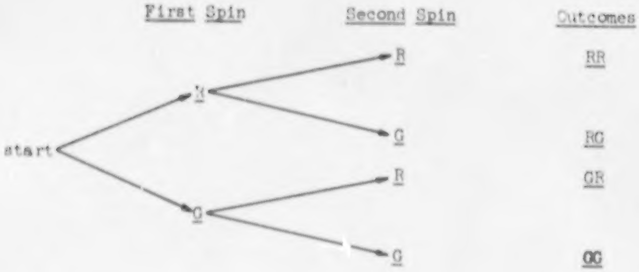
$P(A \cap C) \neq P(A) \cdot P(C)$, hence A and C are not independent events.

3. (a) Answers may vary, but the students by now should feel that the outcome corresponding to the larger area should have a greater number assigned to it, and that $P(G)$ should be $3 \cdot P(R)$. Since $P(G) + P(R)$ must equal 1, we set $P(G) = \frac{3}{4}$ and $P(R) = \frac{1}{4}$.

(b)

		Second Spin	
		R	G
First Spin	R	(R,R)	(R,G)
	G	(G,R)	(G,G)

or



(c) GR has the same probability as RG. It is reasonable to assume that the events "red on first" and "red on second" are independent because the outcome of the first spin has no effect on the outcome of the second spin.

$P(\text{red on first spin}) = \frac{1}{4}$

$P(\text{red on second spin}) = \frac{1}{4}$

$P(RR) = P(R) \cdot P(R) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

(d) $P(RG) = P(R) \cdot P(G) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$

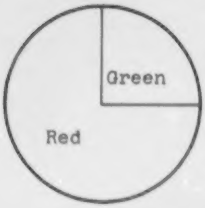
$P(GR) = P(RG) = \frac{3}{16}$

$P(GG) = P(G) \cdot P(G) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

(e) The sum of the probabilities of RR, RG, GR, and GG should be 1.

$P(RR) + P(RG) + P(GR) + P(GG) = \frac{1}{16} + \frac{3}{16} + \frac{3}{16} + \frac{9}{16} = \frac{16}{16} = 1$

4. (a)



The important change is to reverse the colors, since red shows $\frac{3}{4}$ of the time and green shows $\frac{1}{4}$ of the time.

(b) The tree diagram or table for Exercise 3(b) can be used. Note, however, the change in the probabilities assigned.

$P(GG) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ $\qquad$ $P(RG) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$

$P(GR) = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$ $\qquad$ $P(RR) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

(c) $P(GR) = \frac{3}{16}$, $P(GG) = \frac{1}{16}$, so he is 3 times as likely to get a green light in the morning and red in the afternoon as he is to get green lights both times.

(d) $P(RR) = \frac{9}{16}$, $P(GR) + P(RG) + P(GG) = \frac{7}{16}$.

(Or: $P(\text{one green light}) = 1 - P(RR) = 1 - \frac{9}{16} = \frac{7}{16}$.)

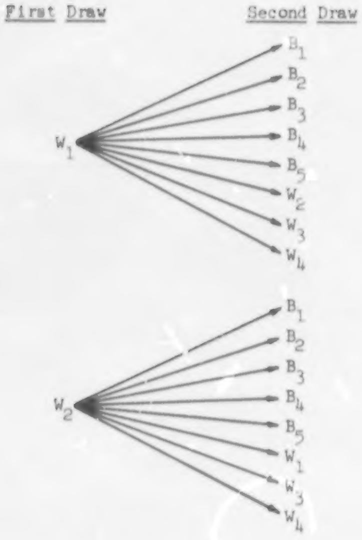
Hence, it is more likely that he will get a red light both morning and afternoon than that he will get at least one green light.

(e) $P(GR) + P(RG) = \frac{6}{16} = \frac{3}{8} = 37\frac{1}{2}\%$. He will get one red and one green light $37\frac{1}{2}\%$ of the time.

5. (a) $P(\text{first marble white}) = \frac{4}{9}$; after replacement,

$P(\text{second marble white}) = \frac{4}{9}$; $\frac{4}{9} \cdot \frac{4}{9} = \frac{16}{81}$.

(b) We may use either a tree diagram or a table to count outcomes. If we use a tree diagram we construct enough to be able to count the outcomes that are of interest for this problem.



Outcomes

W_1B_1
W_1B_2
W_1B_3
W_1B_4
W_1B_5
W_1W_2
W_1W_3
W_1W_4

W_2B_1
W_2B_2
W_2B_3
W_2B_4
W_2B_5
W_2W_1
W_2W_3
W_2W_4

etc.

It becomes apparent that for each of the 9 outcomes of the first draw there are 8 outcomes for the second draw. Hence the total number of outcomes of the succession of two draws is $9 \cdot 8$, or 72.

To compute the number of outcomes in which both marbles are white, note that for each outcome of the first draw which is a white marble, there are 3 white outcomes of the second draw. Since there are 4 white outcomes of the first draw, the number of outcomes of "both white" is $4 \cdot 3 = 12$.

Thus for two draws without replacement, $P(\text{both white}) = \frac{12}{72} = \frac{1}{6}$.

The table would look like this:

Second Draw

First Draw	W_1	W_2	W_3	W_4	B_1	B_2	B_3	B_4	B_5
W_1		W_1W_2	W_1W_3	W_1W_4	W_1B_1	W_1B_2	W_1B_3	W_1B_4	W_1B_5
W_2	W_2W_1		W_2W_3	W_2W_4	W_2B_1	W_2B_2	W_2B_3	W_2B_4	W_2B_5
W_3	W_3W_1	W_3W_2		W_3W_4	W_3B_1	W_3B_2	W_3B_3	W_3B_4	W_3B_5
W_4	W_4W_1	W_4W_2	W_4W_3		W_4B_1	W_4B_2	W_4B_3	W_4B_4	W_4B_5
B_1	B_1W_1	B_1W_2	B_1W_3	B_1W_4		B_1B_2	B_1B_3	B_1B_4	B_1B_5
B_2	B_2W_1	B_2W_2	B_2W_3	B_2W_4	B_2B_1		B_2B_3	B_2B_4	B_2B_5
B_3	B_3W_1	B_3W_2	B_3W_3	B_3W_4	B_3B_1	B_3B_2		B_3B_4	B_3B_5
B_4	B_4W_1	B_4W_2	B_4W_3	B_4W_4	B_4B_1	B_4B_2	B_4B_3		B_4B_5
B_5	B_5W_1	B_5W_2	B_5W_3	B_5W_4	B_5B_1	B_5B_2	B_5B_3	B_5B_4	

In the table it is clear that the main diagonal can have no entries since the second draw is made without replacing the first marble (W_1W_1 is not a possibility). Since each row contains 8 entries and there are 9 rows, the total number of outcomes is $9 \cdot 8 = 72$. The event "two white" is the subset contained in the rectangle in the upper left-hand corner of the table. This rectangle contains 12 entries. Hence P(both white) is $\frac{12}{72} = \frac{1}{6}$.

Either method is acceptable, but the table may be more convincing to students.

[Note to teacher: It is quicker to compute: P(white on first) = $\frac{4}{9}$; P(white on second) = $\frac{3}{8}$; and $\frac{4}{9} \cdot \frac{3}{8} = \frac{1}{6}$. However, this involves "conditional probability", the discussion of which has been postponed.]

6. Taking out two socks is the same as taking out one sock first and then, without replacing it, taking a second. A table is very useful in computing outcomes and the events (subsets) asked for in this problem.

		Second Sock				
		B_1	B_2	B_3	G_1	G_2
First Sock	B_1		B_1B_2	B_1B_3	B_1G_1	B_1G_2
	B_2	B_2B_1		B_2B_3	B_2G_1	B_2G_2
	B_3	B_3B_1	B_3B_2		B_3G_1	B_3G_2
	G_1	G_1B_1	G_1B_2	G_1B_3		G_1G_2
	G_2	G_2B_1	G_2B_2	G_2B_3	G_2G_1	

Since there are 4 entries in each row and there are 5 rows, the total number of outcomes is $5 \cdot 4 = 20$.

(a) The event "both green" is the subset of outcomes in the lower right-hand corner. This rectangle contains two entries, hence $P(\text{both green}) = \frac{2}{20} = \frac{1}{10}$.

(b) "Both blue" occurs in the rectangle in the upper left-hand corner of the table. It contains 6 entries and therefore $P(\text{both blue}) = \frac{6}{20} = \frac{3}{10}$.

(c) P(one green and one blue) is obtained by counting entries in the two remaining parts of the table; since there are 12 entries in these two rectangles, $P(\text{one green and one blue}) = \frac{12}{20} = \frac{3}{5}$.

7. First drawer: $P(\text{black}) = \frac{5}{9}$; $P(\text{blue}) = \frac{4}{9}$.

Second drawer: $P(\text{black}) = \frac{7}{15}$; $P(\text{blue}) = \frac{8}{15}$.

Drawing from the first drawer and drawing from the second drawer are independent events.

(a) $P(\text{both black}) = \frac{5}{9} \cdot \frac{7}{15} = \frac{35}{135} = \frac{7}{27}$

(b) P(both blue) = $\frac{4}{9} \cdot \frac{8}{15} = \frac{32}{135}$

(c) P(one black and one blue) = $1 - (\frac{35}{135} + \frac{32}{135}) = \frac{68}{135}$

Alternatively:

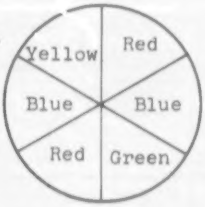
The event "one black and one blue" can occur in two different ways: black from first and blue from second, or
blue from first and black from second.

The probability of the first way is $\frac{5}{9} \cdot \frac{8}{15}$, and the probability of the second way is $\frac{4}{9} \cdot \frac{7}{15}$; so

$$P(\text{one black and one blue}) = \frac{5}{9} \cdot \frac{8}{15} + \frac{4}{9} \cdot \frac{7}{15} = \frac{40}{135} + \frac{28}{135} = \frac{68}{135} .$$

Suggested Test Items

Classify each of the following games as "fair" or "unfair".



1. You spin the spinner at the right. You win if it points to blue or green. You lose if it points to red or yellow.

2. You toss a single die. You win if a prime number is thrown; otherwise you lose.

3. You throw two dice. You win if the sum is 7; you lose if the sum is 6; otherwise there is a tie.

4. You throw two dice. You win if a double is thrown and lose if the sum is 7.

5. You throw two dice. You win if the product of the numbers shown is 18 or greater than 18 and lose if the product is less than 18.

List the set of possible outcomes for each of the following experiments.

6. Throwing one die.

7. Drawing one marble from a bag which contains red, blue, and green marbles.

8. Tossing a penny and a dime.

9. Tossing a single die and then spinning a spinner with red and white regions.

10. Tossing a coin and then drawing a marble from a bag containing blue, green, and yellow marbles.

Find the probability of each of these events.

11. A is the event "number greater than 3" when a single die is tossed. $P(A) = \underline{\ ?\ }$

12. B is the event "black" when a marble is drawn from a bag containing three marbles, one red and two blue. $P(B) = \underline{\ ?\ }$

13. C is the event that a boy will be chosen when a student is selected at random from a class of thirty of whom 18 are girls. $P(C) = \underline{\ ?\ }$

14. D is the event "prime" when a number n such that $40 < n < 50$ is chosen at random. $P(D) = \underline{\ ?\ }$

15. E is the event that an article is up to standard when a sample of 500 has shown 5 defective. $P(E) = \underline{\ ?\ }$

Tell whether each of the following pairs of events are:

(1) mutually exclusive,
(2) complementary,
(3) independent,
(4) none of the above.

If more than one classification applies, list all that apply.

16. A red die and a green die are tossed.
A is the event "2 on the red".
B is the event "3 on the green".

17. One card is drawn from a deck.

C is the event "spade".

D is the event "king".

18. A spinner which is $\frac{1}{4}$ green, $\frac{1}{4}$ red, and $\frac{1}{2}$ blue is spun once.

E is the event "green".

F is the event "red".

19. A die is tossed.

G is the event "number greater than 2".

H is the event "number less than or equal to 2".

20. Two dice are thrown.

I is the event "double".

J is the event "sum is odd".

21. Use the circular regions at the right to draw a spinner such that

(a) P(red) = 40%

(a)

(b) P(black) = .125

(b)

(c) P(green) = 1

(c)

22. On a baseball team, player A has a batting average of .320 and player B has an average of .280. Both come to bat in the seventh inning. Assume "hit for A" and "hit for B" are independent events.

(a) What is the probability that both A and B get hits in the seventh inning?

(b) What is the probability that either A or B or both get hits in the seventh inning?

23. One card is drawn from a hand consisting of the ace and king of hearts and the ace and king of spades.

(a) What is the probability that the card is either an ace or a spade?

(b) What is the probability that the card is either an ace or a king?

Complete:

24. If an event is certain to occur, its probability is ______ .

25. In a probability experiment, events A and B are independent if they satisfy the condition ______.

26. A box of candy contains seven caramels and eleven chocolates. If a piece is chosen at random, P(chocolate) = ______ .

27. If the set of possible outcomes for an experiment consists of four equally likely outcomes, then the probability of each outcome is ______ .

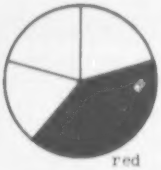
28. The probability of throwing exactly four heads and one tail on a throw of five coins is $\frac{5}{32}$. What is the probability of not throwing four heads and one tail?

29. A stop light shows green $\frac{3}{5}$ of the time. If you make a trip one way in the morning and return in the afternoon, what is the probability of getting a green light going and coming?

30. Suppose you toss a coin five times and each time it shows heads. What is the probability that heads will show on the sixth toss?

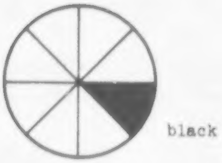
Answers to Suggested Test Items

1. Fair
2. Fair
3. Unfair
4. Fair
5. Unfair
6. {1,2,3,4,5,6,}
7. {red,blue,green}
8. {HH,HT,TH TT}
9. {(1,R),(1,W),(2,R),(2,W),(3,R),(3,W),(4,R),(4,W),(5,R),(5,W),(6,R),(6,W)}
10. {(H,B),(H,G),(H,Y),(T,B),(T,G),(T,Y)}
11. $\frac{1}{2}$
12. 0
13. $\frac{12}{30} = \frac{2}{5}$
14. $\frac{3}{9} = \frac{1}{3}$
15. $\frac{495}{500} = \frac{99}{100}$
16. (3)
17. (4)
18. (1)
19. (1),(2)
20. (1)
21. (a)



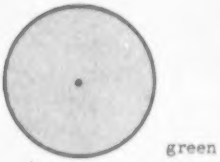
(5 congruent regions)

(b)



$\frac{1}{8}$

(c)



22. (a) $(.320) \cdot (.280) = .0896 \approx .090$

    (b) $.320 + .280 - .090 \approx .510$

23. (a) $P(A) = \frac{1}{2}$; $P(S) = \frac{1}{2}$; $P(A \cap S) = \frac{1}{4}$;

    $P(A \cup S) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$

    (b) $P(A) = \frac{1}{2}$; $P(K) = \frac{1}{2}$; $P(A \cup K) = \frac{1}{2} + \frac{1}{2} = 1$

    Events A and K are mutually exclusive.

24. 1

25. $P(A) \cdot P(B) = P(A \cap B)$

26. $\frac{11}{18}$

27. $\frac{1}{4}$

28. $\frac{27}{32}$

29. $\frac{9}{25}$

30. $\frac{1}{2}$

Tear Sheet (Exercises 10-3b)

2.

	Number on die face					
	1	2	3	4	5	6
Frequency, first row						
Frequency, second row						
Frequency, third row						
Frequency, fourth row						
Total						

3. (a)

Frequency	Like numbers	Consecutive numbers
Exactly two		
Exactly three		
More than three		

4.

Occurrence	Number of blacks
Three of a kind	
Four of a kind	
Five of a kind	